Dynamic Campaign Spending*

Avidit Acharya, Edoardo Grillo, Takuo Sugaya, Eray Turkel†

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Abstract

We build a model of electoral campaigning in which two office-motivated candidates each allocate their budgets over time to affect their relative popularity, which evolves as a mean-reverting stochastic process. We show that in each period, the equilibrium ratio of spending by each candidate equals the ratio of their available budgets. This result holds across different specifications and extensions of the model, including extensions that allow for early voting, and an endogenous budget process. We also characterize how the path of spending over time depends not just on the rate of decay of popularity leads, but also the rate at which returns to spending are diminishing, rates of participation in early voting, and any feedback that short run leads in popularity have on the budget process.

Key words: campaigns, dynamic allocation problems, contests

JEL codes: C72

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†avidit@stanford.edu; edoardo.grillo@carloalberto.org; tsugaya@stanford.edu; eturkel@stanford.edu.

1 Introduction

What factors determine the optimal timing of campaign spending by strategic candidates in the runup to election day? Existing research by Gerber et al. (2011), Hill et al. (2013) and others highlights the decay rate in political advertising, showing that while campaign ads have positive effects on support for the advertising candidate, their effects decay rapidly over time.[1] Given these high decay rates, should candidates spend their resources on political advertising and other persuasion efforts only at the very end of the race? How might we account for early spending?

To answer these questions, we build a simple model in which two candidates, 1 and 2, allocate their resources across time to influence the movement of their relative popularity, and eventually win the election.[2] The candidates begin the game with one being possibly more popular than the other. At each moment in time, relative popularity may go up, meaning that candidate 1's popularity increases relative to candidate 2's; or it may go down. Relative popularity evolves between periods according to a (possibly) mean-reverting Brownian motion, and spending by the candidates affects the drift of this process. At the final date, an election takes place and the more popular candidate wins office. Money left over has no value, so the game is zero-sum.

We show that as long as the drift term (the long-run mean) of the process is quasi-concave and homothetic in the candidates' spending levels, and has an invertible ratio of marginals, then an "equal spending ratio result" holds: at every history, the two candidates spend the same fraction of their remaining budgets. This result rests on the fact that because the game is zero-sum, each candidate faces the same tradeoff in equalizing the marginal return of campaign spending in any period with marginal opportunity cost incurred by not being able to spend the resources later.

We then present an example in which the long-run mean of the process is concave in spending levels so that the returns to spending are diminishing. In this example, the equilibrium ratio of spending by either candidate in consecutive periods is constant over

[1]See DellaVigna and Gentzkow (2010), Kalla and Broockman (2018), Jacobson (2015) and the references in these papers for the state of current knowledge on the effects of political advertising, and persuasion more generally.

[2]A key premise of our model is that advertising can influence elections. For recent evidence on this, see Spenkuch and Toniatti (2018) and Martin (2014). For a summary of prior work on the effects of advertising in elections, see Jacobson (2015).

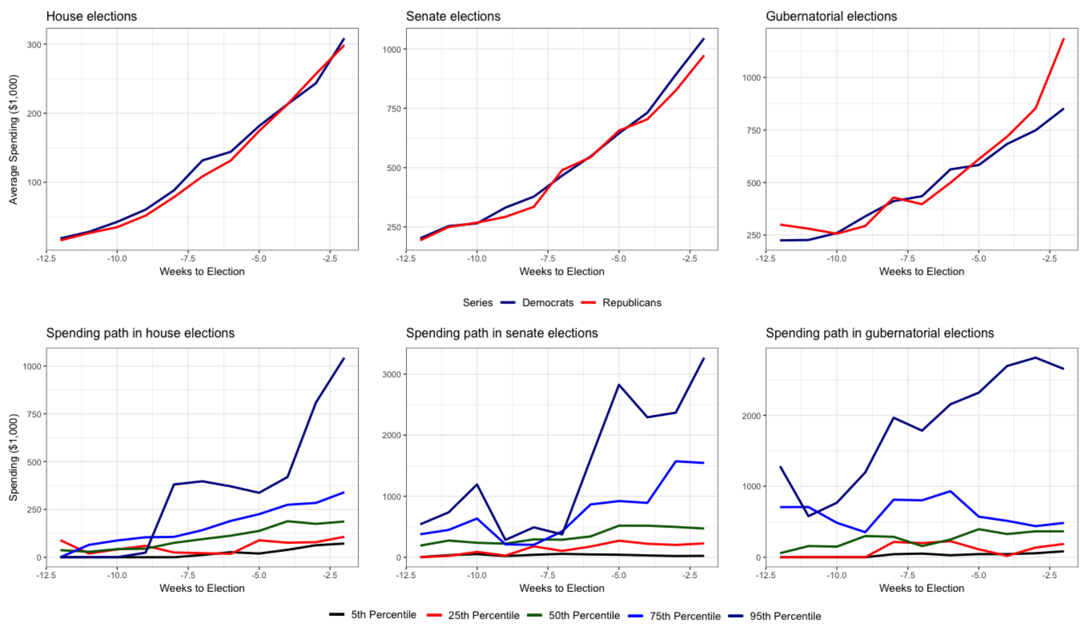

Figure 1: Upper figures are average spending paths by Democrats and Republicans on TV ads in "competitive" House, Senate and gubernatorial races in the period 2000-2014. These are elections in which both candidates spent a positive amount; see Section 5.1 for the source of these data, and more details. Bottom figures are spending paths for 5th, 25th, 50th, 75th, and 95th percentile candidates in terms of total money spent in the corresponding elections of the upper panel.

time, and given by $\exp(-\lambda\Delta/\beta)$, where $\lambda \geq 0$ is the speed of mean reversion of the popularity process, $\Delta > 0$ is the time interval between periods, and $\beta < 0$ measures the rate at which returns are diminishing. When $\lambda = 0$ (the case of no mean-reversion) the candidates spread their resources evenly across periods.[3] When $\lambda > 0$, popularity leads tend to decay between consecutive periods at the rate $1 - e^{-\lambda\Delta}$, and in this case, candidates increase their spending over time. For high values of λ they spend more towards the end of the race and less in the early stages, but λ and β cannot be separately identified from just the spending path.

The fact that spending increases over time when popularity leads tend to decay rationalizes the pattern of spending in actual elections. Figure 1 shows the pattern of TV ad spending over time for candidates in U.S. House, Senate and gubernatorial elections

[3]This special case recovers the finding of a recent paper by Klumpp et al. (2019) that studies a dynamic strategic allocation problem absent the feature of decay, and shows that the equilibrium allocation is constant over time; see our discussion below.

over the period 2000-2014. The upper figures show that the average spending patterns for Democrats and Republicans in these races are nearly identical, and that average spending increases over time. The lower figures show how there is more noise in individual candidates' observed spending, but that the overall pattern of spending growth holds at the individual candidate level as well, especially in those electoral contests that see the highest spending levels.

We also investigate other factors that determine the spending path. We provide an extension in which some voters turn out to vote early, starting several periods prior to the election date. We show that if more voters are expected to cast their ballots early, then candidates save less for the end. We also provide an extension in which the candidates' available budgets evolve over time in response to shifts in relative popularity. In this extension, candidates may have an added incentive to spend early to raise their popularity in the short-term, if these short-term gains helps them raise more resources from donors that they can deploy in the later stages of the race.

Our equal spending ratio result continues to hold in these extensions, and all other settings that we study including one in which the candidates compete in multiple districts or media markets. We therefore end the paper with an examination of the extent to which measurement error and other factors have led to violations of this prediction. Overall, we find that the equal spending ratio result holds up reasonably well: in well above half of the elections, the candidates' spending ratios are within ten percentage points of each other in any given week.

Our paper relates to the prior literature on campaigning, which typically focuses on other aspects of the electoral contest. Kawai and Sunada (2015), for example, build on the work of Erikson and Palfrey (1993, 2000) to estimate a model of fund-raising and campaigning in which the inter-temporal resource allocation decisions that candidates make are across different elections rather than across periods in the run-up to a particular election. de Roos and Sarafidis (2018) explain how candidates that have won past races may enjoy "momentum," which results from a complementarity between prior electoral success and current spending.[4] Meirowitz (2008) studies a static model to show how asymmetries in the cost of effort can explain the incumbency advantage.

[4]Other dynamic models of electoral campaigns in which candidates enjoy momentum—such as Callander (2007), Knight and Schiff (2010), Ali and Kartik (2012)—are models of *sequential voting*.

Polborn and David (2004) and Skaperdas and Grofman (1995) also examine static campaigning models in which candidates choose between positive or negative advertising.[5] Iaryczower et al. (2017) estimate a model in which campaign spending weakens electoral accountability, assuming that the cost of spending is exogenous rather than subject to an inter-temporal budget constraint. Garcia-Jimeno and Yildirim (2017) estimate a dynamic model of campaigning in which candidates decide how to target voters taking into account the strategic role of the media in communication. Finally, Gul and Pesendorfer (2012) study a model of campaigning in which candidates provide information to voters over time, and face the strategic timing decision of when to stop.

Our paper also relates to the literature on dynamic contests (see Konrad et al., 2009, and Vojnović, 2016, for reviews of this literature). In this literature, Gross and Wagner (1950) study a continuous Blotto game; Harris and Vickers (1985, 1987), Klumpp and Polborn (2006) and Konrad and Kovenock (2009) study models of races; and Glazer and Hassin (2000) and Hinnosaar (2018) study sequential contests.

Our paper, in contrast to the above prior work, studies campaigning as a a dynamic strategic allocation problem. In this respect, it relates closely to Klumpp et al. (2019), who also study a dynamic strategic allocation model and find that absent any decay, the allocation of resources over time is constant. Our work builds on theirs by uncovering the fact that the equal spending ratio result holds in a variety of general settings that are motivated by our application to electoral campaigning.

2 Model

Consider the following complete information *dynamic campaigning game* between two candidates, $i = 1, 2$, ahead of an election. Time runs continuously from 0 to T and candidates take actions at times in $\mathcal{T} := \{0, \Delta, 2\Delta, ..., (N-1)\Delta\}$, with $\Delta := T/N$ being the interval between consecutive actions. We identify these times with N discrete

[5] Other static models of campaigning include Prat (2002) and Coate (2004), who investigate how one-shot campaign advertising financed by interest groups can affect elections and voter welfare, and Krasa and Polborn (2010) who study a model in which candidates compete on the level of effort that they apply to different policy areas. Prato and Wolton (2018) study the effects of reputation and partisan imbalances on the electoral outcome.

periods indexed by $n \in \{0, ..., N-1\}$. For all $t \in [0, T]$, we use $\underline{t} := \max\{\tau \in \mathcal{T} : \tau \leq t\}$ to denote the last time that the candidates took actions.

At the start of the game the candidates are endowed with positive resource stocks, $X_0 \geq 0$ and $Y_0 \geq 0$ respectively for candidates 1 and 2.[6] Candidates allocate their resources across periods to influence changes in their relative *popularity.* Relative popularity at time t is measured by a continuous random variable $Z_t \in \mathbb{R}$ whose realization at time t is denoted by z_t. We will interpret this as a measure of candidate 1's lead in the polls. If $z_t > 0$, then candidate 1 is ahead of candidate 2. If $z_t < 0$, then candidate 2 is ahead; and if $z_t = 0$, it is a dead heat. We assume that at the beginning of the game, relative popularity is equal to $z_0 \in \mathbb{R}$.

At any time $t \in \mathcal{T}$, the candidates simultaneously decide how much of their resource stock to invest in influencing their future relative popularity. Candidate 1's investment is denoted x_t while candidate 2's is denoted y_t. The size of the resource stock that is available to candidate 1 at time $t \in \mathcal{T}$ is denoted $X_t = X_0 - \sum_{\tau \in \{t' \in \mathcal{T} : t' < t\}} x_\tau$ and that available to candidate 2 is $Y_t = Y_0 - \sum_{\tau \in \{t' \in \mathcal{T} : t' < t\}} y_\tau$. At every time $t \in \mathcal{T}$, budget constraints must be satisfied, so $x_t \leq X_t$ and $y_t \leq Y_t$.

Throughout, we will maintain the assumption that for all times t, the evolution of popularity is governed by the following Brownian motion:

$$dZ_t = (p(x_{\underline{t}}, y_{\underline{t}}) - \lambda Z_t)\, dt + \sigma dW_t \tag{1}$$

where $\lambda \geq 0$ and $\sigma > 0$ are parameters and $p(\cdot)$ is a twice differentiable real-valued function. Thus, the drift of popularity depends on the candidates' investments through the function $p(\cdot)$. If $\lambda = 0$ the law of motion of relative popularity in the interval between consecutive periods of investment, t and $t+\Delta$, is a standard Brownian motion with drift $p(x_{\underline{t}}, y_{\underline{t}})$. If $\lambda > 0$, it is a mean reverting Brownian motion (the Ornstein-Uhlenbeck process) with long-run mean $p(x_t, y_t)/\lambda$ and speed of reversion λ.

[6] Although candidates raise funds over time, our assumption that they start with a fixed stock is tantamount to assuming that they can forecast how much will be available to them. In fact, some large donors make pledges early on and disburse their funds as they are needed over time. Nevertheless, in Section 4.2 we relax this assumption and consider an extension of the model in which the candidates' resources evolve over time in response to the candidates' relative popularity.

The game ends at time T, with candidate 1 winning if $z_T > 0$, losing if $z_T < 0$, and both candidates winning with equal probability if $z_T = 0$. The winner then collects a payoff of 1 while the loser collects a payoff of 0. Thus, the game is zero sum, and the winner is the candidate that is more popular at time T.

3 Analysis

Since the game is in continuous time, strategies must be measurable with respect to the filtration generated by W_t. However, since candidates take actions only at discrete times, we will forgo this additional formalism and treat the game as a game in discrete time. By our assumption about the popularity process in (1), the distribution of $Z_{t+\Delta}$ at any time $t \in \mathcal{T}$, conditional on (x_t, y_t, z_t), is normal with constant variance and a mean that is a weighted sum of $p(x_t, y_t)$ and z_t; specifically,

$$Z_{t+\Delta} \mid (x_t, y_t, z_t) \sim \begin{cases} \mathcal{N}\left(p(x_t, y_t)\Delta + z_t, \sigma^2\Delta\right) & \text{if } \lambda = 0 \\ \mathcal{N}\left((1 - e^{-\lambda\Delta})p(x_t, y_t)/\lambda + e^{-\lambda\Delta}z_t, \sigma^2(1 - e^{-2\lambda\Delta})/2\lambda\right) & \text{if } \lambda > 0 \end{cases} \tag{2}$$

where $\mathcal{N}(\cdot, \cdot)$ denotes the normal distribution whose first component is mean and second is variance. Note that the mean and variance of $Z_{t+\Delta}$ in the $\lambda = 0$ case correspond to the limits as $\lambda \to 0$ of the mean and variance in the $\lambda > 0$ case.

The model is therefore strategically equivalent to a discrete time model in which relative popularity is a state variable that transitions over discrete periods, and in each period it is normally distributed with a constant variance and a mean that depends on the popularity in the last period and on the candidates' spending levels.

With this, our equilibrium concept is subgame perfect Nash equilibrium (SPE) in pure strategies. We will refer to this concept succinctly as "equilibrium."[7]

3.1 Equal Spending Ratios

The key implication of (2) is that the effect of the spending levels on the next period popularity level is linearly separable from the stochastic terms (Z_t, ε_t), which we can see

[7]Because the game that we consider is zero-sum, for any Nash equilibrium there exists an outcome-equivalent SPE. So we will sometimes look at Nash equilibria to study on-path equilibrium play.

by writing for all $t \in \mathcal{T}$,

$$Z_{t+\Delta} = (1 - e^{-\lambda\Delta})p(x_t, y_t) + e^{-\lambda\Delta} Z_t + \varepsilon_t,$$

where ε_t is a mean-zero normally distributed random variable.[8] By recursive substitution we can write

$$Z_T = (1 - e^{-\lambda\Delta}) \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda N\Delta} + \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}, \quad (3)$$

where $(\varepsilon_\tau)_{\tau \geq 0}$ are i.i.d. normal shocks all with mean 0. Candidate 1 maximizes $\Pr[Z_T > 0]$ and candidate 2 minimizes this probability. Since the coefficient of the ε terms in (3) is independent of all $x_{n\Delta}$ and $y_{n\Delta}$, the variance of Z_T is independent of the candidates' strategies. So we can hereafter write the objective of candidate 1 as maximizing the expected value of Z_T and the objective of candidate 2 as minimizing it.

We say that an equilibrium is *interior* if the first order conditions for these maximization problems are satisfied at the equilibrium. From the expression above, we see that a unique equilibrium exists if $p(\cdot, y)$ is quasiconcave for all y and $p(x, \cdot)$ is quasiconvex for all x, and the equilibrium is interior. Moreover, the spending profile (x_t, y_t) is notably independent of z_t.

We will also maintain the assumption, throughout the paper, that p is a homothetic function with an invertible ratio of marginals; specifically—

Assumption A. *There is an invertible function $\psi : (0, \infty) \to \mathbb{R}$ s.t.*

$$\forall x, y > 0, \quad \frac{p_x(x, y)}{p_y(x, y)} = \psi(x/y).$$

We refer to the ratio of a candidate's current spending to current budget as that candidate's *spending ratio*. For candidate 1 this is x_t/X_t and for candidate 2 it is y_t/Y_t. The following theorem summarizes the key observations that we have made so far, and shows that if Assumption A holds, then in equilibrium the two candidates' spending ratios equal each other. This is our "equal spending ratio result."

[8]Using the result in Karatzas and Shreve (1998) equation (6.30), we can write down sufficient conditions to obtain this separability. Details are available upon request.

Theorem 1. *There exists a unique equilibrium if $p(\cdot, y)$ is quasiconcave in all y and $p(x, \cdot)$ is quasiconvex in all x, and the equilibrium is interior. In the equilibrium, x_t/X_t and y_t/Y_t are independent of the past history $(z_\tau)_{\tau \le t}$ of relative popularity. Under Assumption A, the equal spending ratio result also holds; that is, in equilibrium,*

$$x_t/X_t = y_t/Y_t \text{ for all } t \in \mathcal{T} \text{ s.t. } X_t, Y_t > 0.$$

3.2 An Example

Assumption A is satisfied, for example, by $p(x, y) = h(\alpha_1\varphi(x) - \alpha_2\varphi(y))$ where h is a twice differentiable function, α_1 and α_2 are constants, and φ is a function such that $\varphi'(x) = x^\beta$ for some parameter β.[9] This provides a parametric example provided that the function h and parameters α_1, α_2 and β are chosen so that the quasi-concavity assumptions hold.[10] Given Z_T from (3), at any time $t \in \mathcal{T}$ candidate 1 maximizes $\Pr[Z_T \ge 0 \mid z_t, X_t, Y_t]$ under the constraint $\sum_{n=t/\Delta}^{N-1} x_{n\Delta} \le X_t$, while candidate 2 minimizes this probability under the constraint $\sum_{n=t/\Delta}^{N-1} y_{n\Delta} \le Y_t$. Using this fact, we can apply the Euler method from consumer theory to solve the equilibrium, provided the first order conditions are sufficient and h is a homogenous function of degree 1.[11]

The candidates' first order conditions with respect to $x_{n\Delta}$ and $y_{n\Delta}$ for each $n < N-1$ are respectively

$$e^{-\lambda\Delta(N-1-n)} x_{n\Delta}^\beta h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) = x_{(N-1)\Delta}^\beta h'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta}))$$
$$e^{-\lambda\Delta(N-1-n)} y_{n\Delta}^\beta h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) = y_{(N-1)\Delta}^\beta h'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta}))$$

Note that we can recover the equal spending ratio result from taking the ratios of these conditions. The equal spending ratio result then implies that the ratio of spending in

[9] The assumption holds, defining $\psi(x/y) = -(\alpha_1/\alpha_2)(x/y)^\beta$.

[10] If h is the identity, for example, the assumptions needed for an interior equilibrium are satisfied for $\beta < 0$ and $\alpha_1, \alpha_2 > 0$.

[11] Actually, provided that the first order conditions are sufficient, we can even let h be homogenous of degree d for arbitrary $d \ge 1$. In this case, the result of Proposition 1 below will hold with

$$r = \exp\left(-\frac{\lambda\Delta}{(1+\beta)d - 1}\right).$$

consecutive periods r_n is the same on the equilibrium path for both candidates; i.e.

$$r_n := \frac{x_{(n+1)\Delta}}{x_{n\Delta}} = \frac{y_{(n+1)\Delta}}{y_{n\Delta}}$$

To find the equilibrium, we will guess that the rate of spending growth is constant over time, i.e., $r_n = r$ for all n, and then verify this guess.

If we equate the left hand sides of candidate 1's first order conditions for two consecutive periods n and $n+1$ we get

$$e^{-\lambda\Delta} x_{n\Delta}^{\beta} h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) = x_{(n+1)\Delta}^{\beta} h'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) \tag{4}$$

If the guess of constant spending growth is correct then

$$\begin{aligned} h'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) &= h'(r^{1+\beta}(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta}))) \\ &= h'((\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta}))) \end{aligned}$$

since $\varphi(x) = x^{1+\beta}/(1+\beta)$ and the derivative of a homogenous function of degree 1 is a homogenous function of of degree 0. Therefore, using this in equation (4), we get that $r = \exp(-\lambda\Delta/\beta)$. The same holds for candidate 2. This verifies our guess that the consecutive period spending ratio is constant over time; we refer to this as the "constant spending growth result." The proposition below summarizes our findings.

Proposition 1. *In the equilibrium of the example above, the consecutive period spending ratio is, for all n,*

$$r_n = r = \exp\left(-\frac{\lambda\Delta}{\beta}\right)$$

With fixed budgets, the ratio of consecutive period spending is sufficient to characterize the path of spending over time. For example, consider a benchmark case where h is the identity, and $\alpha_1, \alpha_2, -\beta > 0$ so that the assumptions for an interior equilibrium are satisfied. If $\lambda = 0$, meaning that popularity leads do not decay, then $r = 1$ and the candidates spend their resources equally across periods, spending a fraction $1/N$ of their budget each period. If $\lambda > 0$ then spending increases over time, and the fraction

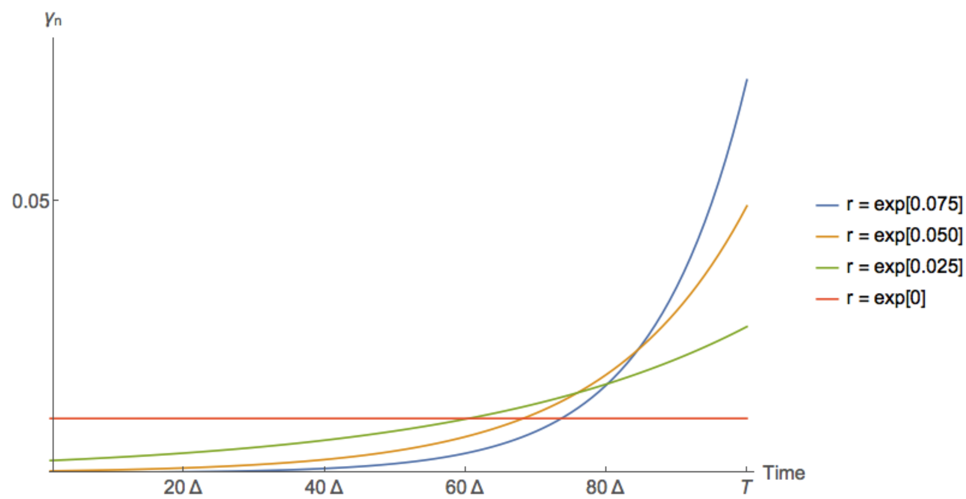

Figure 2: The fraction γ_n of initial budget that the candidates spend over time, for $N = 100$ and various values of r.

of their initial budget that each candidate spends in period n is

$$\gamma_n = \frac{x_n}{X_0} = \frac{y_n}{Y_0} = \frac{r-1}{r^N - 1} r^n$$

where r is the constant ratio of spending in consecutive periods. Substituting r from Proposition 1, we can derive the comparative statics of γ_n with respect to the parameters. If β increases, the marginal return to spending diminishes at a slower rate, providing candidates with less incentive to smooth their spending over time; so they spend more towards the end.[12] As λ increases, the marginal benefit of spending early drops since any popularity advantage produced by an early investment has a tendency to decay, and this tendency is greater for higher values of λ. This means that the candidates have an incentive to invest less in the early stages and more in the later stages of the race.

Figure 2 depicts these features by plotting γ_n for different values of r.

Remark 1. Several robustness results follow directly from the fact that our game is zero sum. First, proof of Theorem 1 in the appendix actually shows that the Nash equilibrium

[12]In fact, as $\beta \to 0^-$ the marginal return to spending does not diminish and candidates spend all of their resources in the final period.

of the game is unique. Second, since the unique equilibrium is in pure strategies, the results are also robust to having the candidates move sequentially within a period, with arbitrary (and possibly stochastic) order of moves across periods. Finally, the results are also robust to allowing the final payoffs to depend linearly on Z_T (an assumption that encompasses the case where candidates care not just about winning but also about margin of victory) so long as the game remains zero-sum.

Remark 2. Since the equilibrium strategies do not depend on realizations of the relative popularity path, results are also robust to having the candidates imperfectly and asymmetrically observe the realization of the path of popularity.

Remark 3. The fact that spending is independent of the past history of relative popularity is implied by the linear separability of $p(x, y)$ from the stochastic part of the process. However, this separability is not necessary for our equal spending ratio result. In particular, suppose that the popularity process is

$$dZ_t = (p(x_{\underline{t}}, y_{\underline{t}}, z_{\underline{t}}) - \lambda Z_t)dt + \sigma dW_t$$

rather than (1), but that Assumption A continues to hold in the sense that there is an invertible function ψ such that for all $t \in \mathcal{T}$, if $x, y > 0$ then $p_x(x, y, z)/p_y(x, y, z) = \psi(x/y)$. Then the linear separability of the investment effect from stochastic terms no longer holds. A special case of this arises if there are functions q, ζ such that $p_x(x, y, z) = q_x(x, y)\zeta(z)$ and $p_y(x, y, z) = q_y(x, y)\zeta(z)$ and q is a homothetic function with invertible ratio of marginals. The same proof of Theorem 1 in the appendix shows that the equal spending ratio result holds; however, the spending path will in general depend on the popularity process through the function ζ.

Similarly, if $p(x, y, z)$ depends on x, y only through the ratio x/y (e.g., $p(x, y, z) = h(x/y, z)$ for some continuous and strictly concave function h), then Assumption A holds and therefore the equal spending ratio result also holds.[13] Indeed, if $(x^*_\tau, y^*_\tau)_{\tau \geq t}$ is an

[13] For this case, the description of the model is not closed since p is undefined if $y = 0$. To get around this, we make the assumption that if either candidate i spends 0 at any time in $\mathcal{T}$, then the game ends immediately. If candidate $j \neq i$ spends a positive amount at that time, then j is the winner while if j also spends 0 at that time, then each candidate wins with probability 1/2. This ensures that Z_t will follow an Itô process at every history, and the model can be considered the limiting case of two different models. One is a model in which the marginal return to spending an ϵ amount of resources

equilibrium in the continuation game in which the candidates' remaining budgets are $X_t, Y_t > 0$ then $(\theta x_\tau^*, \theta y_\tau^*)_{\tau \geq t}$ must be an equilibrium when the budgets are $\theta X_t, \theta Y_t$, for all $\theta > 0$.[14] This observation serves as the basis for our extensions to the case of an endogenous budget process that we develop in Section 4.2 below.

Remark 4. In addition to the decision of when to spend, candidates also make decisions about where to spend their money. Suppose the candidates compete in S winner-take-all districts or media markets (rather than a single district, or market) and whether or not a candidate wins depends on how these individual contests aggregate.[15] Relative popularity in each district s is the random variable Z_t^s with realizations z_t^s, and we assume that the joint distribution of the vector $(Z_{t+1}^s)_{s=1}^S$ depends on $(x_t^s/y_t^s, z_t^s)_{s=1}^S$ only. This allows for arbitrary correlation of relative popularity across districts. If all other structural features are the same as in the baseline model, then in equilibrium the equal spending ratio result holds district-by-district: if $X_t, Y_t > 0$ are the remaining budgets of candidates 1 and 2 at any time $t \in \mathcal{T}$, then $x_t^s/X_t = y_t^s/Y_t$ for all districts s. (See Appendix A.2 for the details.) The key implication of this result is that the total spending of each of the two candidates across all districts at a given time also respects the equal spending ratio result: if $x_t = \sum_s x_t^s$ is candidate 1's total spending at time t and $y_t = \sum_s y_t^s$ is candidate 2's, then $x_t/X_t = y_t/Y_t$, for all $t \in \mathcal{T}$.

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starting at 0 goes to infinity. The other is one in which candidates have to spend a minimum amount ϵ in each period to sustain the campaign, and ϵ goes to 0. However, with these assumptions we will have to say that equilibria are "essentially" unique since there is a trivial source of multiplicity that arises at histories in which one candidate spends 0 in a given period (though these histories do not arise on the equilibrium path). In this case, if the other candidate has a positive resource stock, he may spend any positive amount in that period and win. Apart from this kind of multiplicity, equilibria will be is unique.

[14] If this were not the case, we could find $(\tilde{x}_\tau)_{\tau \geq t}$ that gives candidate 1 a higher probability of winning given $(\theta y_\tau^*)_{\tau \geq t}$. Because Z_T is determined by $(x_\tau/y_\tau)_{\tau \geq t}$, this would imply that the distribution of Z_T given $(\tilde{x}_\tau/\theta y_\tau^*)_{\tau \geq t}$ is more favorable to candidate 1 than the distribution given $(\theta x_\tau^*/\theta y_\tau^*)_{\tau \geq t} = (x_\tau^*/y_\tau^*)_{\tau \geq t}$. Because $(\tilde{x}_\tau/\theta)_{\tau \geq t}$ and $(y_\tau^*)_{\tau \geq t}$ are feasible continuation spending paths when the budgets are (X_t, Y_t) , this would contradict the optimality of $(x_\tau^*)_{\tau \geq t}$ when candidate 2 spends $(y_\tau^*)_{\tau \geq t}$.

[15] Since we can assume an arbitrary aggregation rule, this setting is general enough to cover the electoral college for U.S. presidential elections, as well as competition between two parties seeking to control a majoritarian legislature composed of representatives from winner-take-all single-member districts, and even the case where candidates compete in a single winner-take-all race but must choose how to allocate spending across different geographic media markets in the district.

4 Extensions

4.1 Early Voting

In many elections, voters are able to cast their votes prior to election day, either by mail or in person. Our model is able to accommodate this kind of early voting.

Suppose that all other features of the example in Section 3.2 continue to hold, but now voters can vote early from time $\hat{N}\Delta$ onwards, where $\hat{N} < N$ is an integer. Furthermore, suppose that the vote difference among votes cast for each candidate starting from period $\hat{N}\Delta$ is proportional in each period $n \geq \hat{N}$ to that period's relative popularity $Z_{n\Delta}$. Finally, let the number of total votes cast in period $n \geq \hat{N}$ be a proportion $\xi \in (0, 1)$ of the total votes cast in period $n + 1$. Therefore, the higher is ξ, the lower is the growth rate in votes cast as election day approaches, and if ξ is close to zero, then almost all votes are cast at time T. Then, the objective is thus for candidate 1 to maximize (and candidate 2 to minimize):[16]

$$\Pr\left\{\sum_{k=0}^{N-\hat{N}} \xi^k Z_{(N-k)\Delta} \geq 0\right\}$$

Proposition 2. *In the equilibrium of this early voting extension, the equal spending ratio result holds: if $X_t, Y_t > 0$ then $x_t/X_t = y_t/Y_t$ for all $t \in \mathcal{T}$. In addition, the consecutive period spending ratio for both candidates is*

$$\hat{r}_n = \begin{cases} e^{-\frac{\lambda\Delta}{\beta}} & \text{if } n < \hat{N} \\ K(\xi, \lambda\Delta)e^{-\frac{\lambda\Delta}{\beta}} & \text{if } n \geq \hat{N} \end{cases}$$

[16]This objective function implicitly assumes that, despite early voting, either candidate can win the election if his popularity at time T is sufficiently high, no matter how low it was in previous periods. This holds if $\xi(2 - \xi^{N-\hat{N}}) < 1$, which is implied by $\xi < 1/2$. Alternatively, the results of Proposition 2 would hold if we assume that candidate 1 maximizes (and candidate 2 minimizes) the difference in candidate 1 and 2's vote share, which we could write as being

$$\sum_{k=0}^{N-\hat{N}} \xi^k Z_{(N-k)\Delta}$$

where
$$K(\xi, \lambda\Delta) := \left(\frac{(e^{-\lambda\Delta}/\xi) - (e^{-\lambda\Delta}/\xi)^{N-n-1}}{1 - (e^{-\lambda\Delta}/\xi)^{N-n-1}} \right)^{-\frac{1}{\beta}}$$
which is lower than 1, decreasing in n *and decreasing in* ξ*.*

Besides establishing the equal spending ratio result in this setting, Proposition 2 has two main implications. First, with early voting, the consecutive period spending ratio is no longer constant over time. Second, as early voting turnout rates increase (i.e., ξ increases), spending patterns become more evenly distributed over time. In particular, this extension highlights that with early voting, the dynamic pattern of spending is determined by two countervailing forces: the decay rate popularity leads, measured by λ, leads candidates to spend more resources toward the end of the race, while early voting turnout, measured by ξ, gives the candidates an added incentive to spend more in the earlier stages of the race.

4.2 Evolving Budgets

Our baseline model assumes that candidates are endowed with a fixed budget at the start of the game (or they can perfectly forecast how much money they will raise), but in reality the amount of money raised may depend on how well the candidates poll over the campaign cycle. To account for this, we present an extension here in which the resources stock also evolves in a way that depends on the evolution of popularity. We retain all the features of the baseline model except the ones described below.

Candidates start with exogenous budgets X_0 and Y_0 as in the baseline model. However, we now assume that the budgets evolve according to the following geometric Brownian motions:

$$\frac{dX_t}{X_t} = az_t dt + \sigma_X dW_t^X \quad \text{if } X_t > 0$$
$$\frac{dY_t}{Y_t} = bz_t dt + \sigma_Y dW_t^Y \quad \text{if } Y_t > 0$$

where a, b, σ_X and σ_Y are constants, and W_t^X and W_t^Y are Wiener processes. None of our results hinge on it, but we also make the assumption for simplicity that dW_t is independent of dW_t^X and of dW_t^Y, while dW_t^X and dW_t^Y have covariance $\rho \geq 0$. If either of the two budgets reaches 0 at a given moment in time, it is 0 thereafter.[17]

In this setting, if $b < 0 < a$ then donors raise their support for candidate that is leading in the polls and withdraw support from the one that is trailing. If $a < 0 < b$ then donors channel their resources to the underdog. Popularity therefore feeds back into the budget process. The feedback is positive if $a - b > 0$ and negative if $a - b < 0$. We refer to a and b as the feedback parameters.[18]

All other features of the model are exactly the same as in the baseline model, including the process (1) governing the evolution of popularity, though we now assume for analytical tractability that[19]

$$p(x, y) = \log(x/y).$$

Proposition 3. *In the model with evolving budgets, for every N, T, and $\lambda > 0$, there exists $-\eta < 0$ such that whenever $a - b \geq -\eta$, there is an essentially unique equilibrium. For all $t \in \mathcal{T}$, if $X_t, Y_t > 0$, then in equilibrium,*

$$x_t/X_t = y_t/Y_t.$$

To understand the condition $a - b \geq -\eta$, note that when $a < 0 < b$, there is a negative feedback between popularity and the budget flow: a candidate's budget increases when she is less popular than her opponent. The condition $a - b \geq -\eta$ puts a bound on how negative this feedback can be. If this condition is not satisfied, candidates may want to *reduce* their popularity as much as they can in the early stages of the campaign to accumulate a larger war chest to use in the later stages. This could undermine the existence of an equilibrium in pure strategies.

One question that we can ask for this extension is how the distribution of spending over time varies with the feedback parameters a and b that determine the rate of flow of candidates' budgets in response to shifts in relative popularity. In the baseline model, when $\lambda > 0$ the difficulty in maintaining an early lead means that there is a disincentive

[17]Formally, if $X_t = 0$ ($Y_t = 0$), then $X_\tau \equiv 0$ ($Y_\tau \equiv 0$) for all $\tau \geq t$.

[18]Also, note that dX_t and dY_t may be negative. One interpretation is that X_t and Y_t are expected total budgets available for the remainder of the campaign, where the expectation is formed at time t. Depending on the level of relative popularity, the candidates revise their expected future inflow of funds and adjust their spending choices accordingly.

[19]See our remark in footnote 13 regarding this assumption.

to spend resources early on. This produces the result that spending is increasing over time. However, in this extension, if $b < 0 < a$ then there is a force working in the other direction: spending to build early leads may be advantageous because it results in faster growth of the war chest, which is valuable for the future. The disincentive to spend early is mitigated by this opposing force, and may even be overturned if a is much larger than b, i.e., if donors have a greater tendency to flock to the leading candidate.

We can establish this intuition formally. Recall that r_n defined for the example in Section 3.2 above gave the ratio of equilibrium spending in consecutive periods, n and $n+1$. For this extension with evolving budgets, we define the analogous ratio which we show in the appendix is the same for both candidates:

$$\tilde{r}_n = \frac{x_{(n+1)\Delta}/X_{(n+1)\Delta}}{x_{n\Delta}/X_{n\Delta}} = \frac{y_{(n+1)\Delta}/Y_{(n+1)\Delta}}{y_{n\Delta}/Y_{n\Delta}}$$

We also show in the appendix that this ratio depends on the budget feedback parameters, a and b, only through the difference $a - b$.

Proposition 4. *Fix the number of periods N, total time $T = N\Delta$, and consider the case in which $\lambda > 0$. Then, in equilibrium, for all n, if $a - b$ is sufficiently small then the consecutive period spending ratio $\tilde{r}_n$ conditional on the history up to period n is (i) greater than 1, (ii) increasing in λ, and (iii) decreasing in $a - b$.*

The baseline model with $p(x, y) = \log(x/y)$ is a special case of this model with evolving budgets in which the total budget is constant over time: $a = b = \sigma_X = \sigma_Y = 0$. What Proposition 4 says is that starting with this special case, as we increase the difference $a - b$ from zero, spending plans becomes more balanced over time: there is a greater incentive to spend in earlier periods of the race than there is if $a = b$.[20]

[20] The results of Proposition 4 do not necessarily hold when $a - b$ is very large. We have examples in which $\tilde{r}_n$ is *increasing* in $a - b$ for large λ, n, and $a - b$. (One such example is $\lambda = 0.8$, $\Delta = 0.9$, and $n = a - b = 10$.) The intuition behind these examples rests on the fact that when the degree of mean reversion is high, then it is important for candidates to build up a large war chest that they can deploy in the final stages of the race. If the election date is distant and $a - b$ is large, then early spending is mostly for the purpose of building up these resources. But spending too much in any one period, especially an early period, is risky: if the resource stock does not grow (or even if it grows but insufficiently) then there is less money, and hence not much opportunity, to grow it in the subsequent periods. Since p is concave, the candidates would like to have many attempts to grow the war chest early on, and this is even more the case as the importance of the relative feedback $a - b$ gets large.

Remark 5. The popularity process can feed back into candidates' budgets also in other ways. For example, contributions may be higher when the race is close ($|z_t|$ is small), and lower when one of the candidates has a solid lead ($|z_t|$ is large). To capture this possibility, we can modify the budgets' laws of motion as follows:

$$\frac{dX_t}{X_t} = \frac{a}{1+z_t^2}dt + \sigma_X dW_t^X \quad \text{and} \quad \frac{dY_t}{Y_t} = \frac{b}{1+z_t^2}dt + \sigma_Y dW_t^Y,$$

with $a, b > 0$. Proposition A.3 of Appendix A.6 shows that the equal spending ratio result continues to hold in this case. However, in this setting closed-form characterizations of the spending path cannot be obtained in general. This is because the drifts of the budget processes depend non-linearly on popularity. However, one special case in which closed-form solutions can be obtained occurs when $a = b$. Under this assumption, the percentage change in campaign budgets arising from movements in relative popularity is the same for both candidates. As a result, the interior equilibrium is essentially unique and its closed form expression is identical to the one derived in Proposition A.2 of Appendix A.4, for the special case in which $a = b$.

5 Descriptive Data

The main robust prediction of our model is the equal spending ratio result. Figure 1 in the introduction reveals that there are some violations of this prediction in the data, which are in part due to noisy observations of the actual spending path, but may also be driven by factors not captured in our model. We now offer descriptive look at actual electoral spending data to examine the extent to which this prediction is actually violated. In Appendix B we also look at the extent to which the constant spending growth result of the example in Section 3.2 appears to be violated.

5.1 Data

We focus on subnational American elections, namely U.S. House, Senate, and gubernatorial elections in the period 2000 to 2014.

Spending in our model refers to all spending—TV ads, calls, mailers, door-to-door canvasing visits—that directly affects the candidates' relative popularity. But for some of these categories of spending, it is not straightforward to separate out the part of spending that has a direct impact on relative popularity from the part that does not (e.g. fixed administrative costs). For one category, namely TV advertising, it is straightforward to do this, so we focus exclusively on TV ad spending. Television advertising constitutes around 35% of the total expenditures by congressional candidates, and is approximately 90% of all advertising expenditure (Albert, 2017). We proceed under the assumption that any residual spending on other types of campaign activities that directly affect relative popularity is proportional to spending on TV ads.

Our TV ad spending data are from the Wesleyan Media Project and the Wisconsin Advertising Database. For each election in which TV ads were bought, the database contains information about the candidate each ad supports, the date it was aired, and the estimated cost. For the year 2000, the data covers only the 75 largest Designated Market Areas (DMAs), and for years 2002-2004, it covers only the 100 largest DMAs. The data from 2006 onwards covers all of the 210 DMAs. We obtain the amount spent on ads from total ads bought and price per ad. For 2006, where ad price data are missing, we estimate prices using ad prices in 2008.[21]

We focus on races where the leading two candidates in terms of vote share are from the Democratic and the Republican party. We label the Democratic candidate as candidate 1 and the Republican candidate as candidate 2, so that x_t, X_0, etc. refer to the Democrat's spending, budget, etc. and y_t, Y_0, etc. refer to the Republican's.

We aggregate ad spending made on behalf of the two major parties' candidates by week and focus on the twelve weeks leading to election day, though we will drop the final week which is typically incomplete since elections are held on Tuesdays. We then drop all elections that are clearly not genuine contests to which our model does not apply, defining these to be elections in which one of the candidates did not spend anything for

[21] One concern with this approach could be that if prices increase as the election approaches, then the increase in total spending over time confounds the price increase with increased advertising. However, federal regulations limit the ability of TV stations to increase ad prices as the election approaches, and instead requires them to charge political candidates "the lowest unit charge of the station for the same class and amount of time for the same period" (Chapter 5 of Title 47 of the United States Code 315, Subchapter III, Part 1, Section 315, 1934). This fact allays some of this concern.

Table 1: Descriptive Statistics

	N	Open Seat Election	Incumbent Competing	No Excuse Early Voting	Average total spending	Average spending difference
Senate	122	68	54	82	6019 (5627)	1962 (2921)
Governor	133	59	74	92	5980 (9254)	3173 (6,337)
House	346	97	249	223	1533 (1304)	521 (615)
Overall	601	224	377	397	3428 (5581)	1401 (3,461)

Week	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
Senate											
Avg. spending	196	250	266	314	357	477	545	652	716	860	1,002
	(291)	(328)	(403)	(487)	(401)	(505)	(577)	(724)	(803)	(947)	(1,047)
% spending 0	0.270	0.180	0.123	0.082	0.008	0	0	0	0	0	0
Governor											
Avg. spending	262	253	258	316	420	416	530	597	701	800	1,019
	(632)	(468)	(424)	(581)	(865)	(579)	(1,249)	(1,015)	(1,305)	(1,523)	(1,956)
% spending 0	0.297	0.207	0.139	0.068	0.030	0	0	0	0	0	0
House											
Avg. spending	17	27	38	56	83	120	137	177	212	250	303
	(41)	(55)	(57)	(85)	(93)	(134)	(134)	(182)	(219)	(270)	(340)
% spending 0	0.653	0.545	0.386	0.246	0.095	0	0	0	0	0	0

Note: The upper panel reports the breakdown of elections that are open seat versus those that have an incumbent running, the numbers in which voters can vote early without an excuse to do so, average spending levels by the candidates, and the average difference in spending between the two candidates, all by election type. The lower panel presents average spending for each week in our dataset, and the percent of candidates spending 0 in each week, all by election type. Standard deviations for averages are reported in parentheses. All monetary amounts are in units of $1,000.

more than half of the period studied. This leaves us with 346 House, 122 Senate, and 133 gubernatorial elections.[22] We focus on the last twelve weeks mainly because we want to restrict attention to the general election campaign, and we define the total budgets of the candidates to be the total amount that they spent over these twelve weeks.[23] Summary statistics for spending are given in Table 1. On average, candidates spent about $6 million on TV ads for statewide races, and $1.5 million for House races. There

[22]A tabulation of these elections is given in the Appendix.

[23]In some elections, the primaries end more than twenty weeks prior to the general election date, but ad spending in the period prior to twelve weeks from the election date is typically zero, anyway.

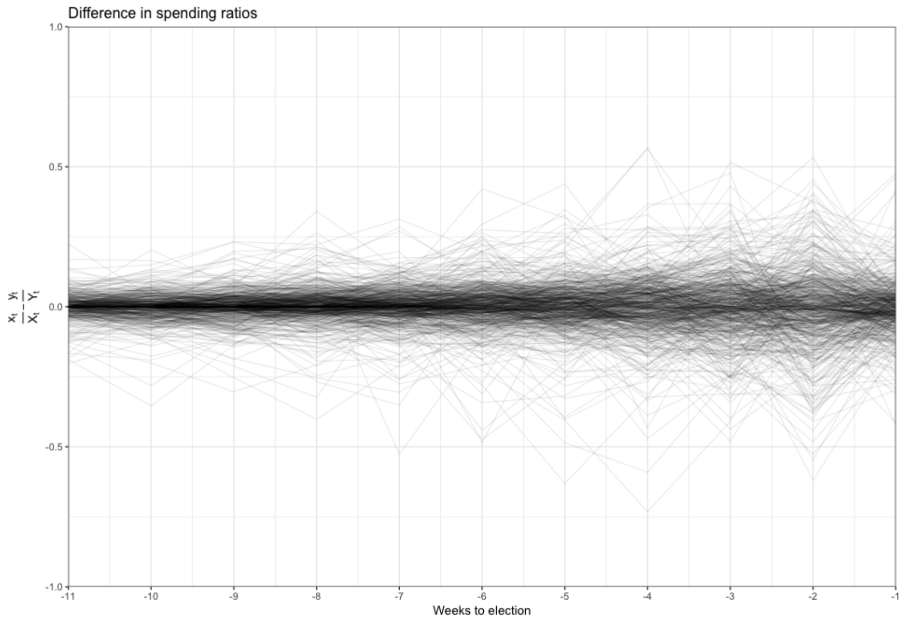

Figure 3: The difference in spending ratios between the Democratic candidate (x_t/X_t) and the Republican candidate (y_t/Y_t) for each week in our dataset. Each line is an election.

is considerable difference in the amount and pattern of spending between state-wide and House elections, so we proceed in analyzing the data using this disaggregation.

5.2 Examining the Equal Spending Ratio Result in the Data

To investigate the extent to which the equal spending ratio result holds in the data, we plot the difference $x_t/X_t - y_t/X_t$ over the final twelve weeks of each election in Figure 3 and tabulate the percent of elections, by election type, in which each candidate's spending was within 10 and 5 percentage points of the other's in Table 2.[24] Overall, Table 2 shows that the prediction seems to be violated to a smaller extent in statewide

[24]Note that since these values are defined as the share of *remaining* budget rather than *total* budget, they can take any value between 0 and 1 in every week in the data prior to the final week. (For example, a candidate can be spending 99% of their remaining budget in every week until the final week.) In the

Table 2: $x_t/X_t - y_t/Y_t$

Week	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
% ∈ (-0.1,0.1)	0.963	0.953	0.938	0.902	0.879	0.847	0.829	0.754	0.676	0.622	0.797
Senate	0.943	0.934	0.975	0.926	0.934	0.885	0.844	0.787	0.746	0.648	0.803
Governor	0.932	0.910	0.887	0.820	0.812	0.812	0.767	0.774	0.639	0.624	0.782
House	0.983	0.977	0.945	0.925	0.884	0.847	0.847	0.734	0.665	0.613	0.801
Early Voting	0.970	0.955	0.942	0.912	0.884	0.844	0.816	0.753	0.673	0.612	0.798
No Early Voting	0.951	0.951	0.931	0.882	0.868	0.853	0.853	0.755	0.681	0.642	0.794
Open Seat	0.942	0.933	0.920	0.897	0.857	0.862	0.866	0.795	0.705	0.656	0.804
Incumbent Competing	0.976	0.966	0.950	0.905	0.891	0.838	0.806	0.729	0.658	0.602	0.793
Close Election	0.976	0.965	0.935	0.941	0.947	0.924	0.906	0.882	0.776	0.706	0.788
Not Close Election	0.958	0.949	0.940	0.886	0.852	0.817	0.798	0.703	0.636	0.589	0.800
Close Budgets	0.974	0.974	0.959	0.925	0.914	0.895	0.883	0.812	0.763	0.695	0.838
Not Close Budgets	0.955	0.937	0.922	0.884	0.851	0.809	0.785	0.707	0.606	0.564	0.764
% ∈ (-0.05,0.05)	0.865	0.815	0.757	0.727	0.661	0.599	0.554	0.468	0.418	0.369	0.562
Senate	0.811	0.762	0.664	0.762	0.713	0.648	0.639	0.566	0.492	0.393	0.598
Governor	0.782	0.744	0.759	0.639	0.586	0.519	0.489	0.481	0.406	0.346	0.556
House	0.916	0.861	0.789	0.749	0.671	0.613	0.549	0.428	0.396	0.370	0.552
Early Voting	0.864	0.814	0.763	0.746	0.660	0.602	0.542	0.463	0.406	0.370	0.562
No Early Voting	0.868	0.819	0.745	0.691	0.662	0.593	0.578	0.475	0.441	0.368	0.564
Open Seat	0.799	0.799	0.723	0.763	0.688	0.625	0.603	0.549	0.460	0.362	0.580
Incumbent Competing	0.905	0.825	0.777	0.706	0.645	0.584	0.525	0.419	0.393	0.374	0.552
Close Election	0.853	0.841	0.841	0.812	0.741	0.729	0.706	0.553	0.535	0.424	0.576
Not Close Election	0.870	0.805	0.724	0.694	0.629	0.548	0.494	0.434	0.371	0.348	0.557
Close Budgets	0.880	0.842	0.789	0.793	0.741	0.677	0.628	0.526	0.515	0.474	0.590
Not Close Budgets	0.854	0.794	0.731	0.675	0.597	0.537	0.496	0.421	0.340	0.287	0.540
Average x_t/X_t	0.021	0.028	0.039	0.054	0.075	0.109	0.134	0.184	0.251	0.377	0.728
	(0.032)	(0.036)	(0.044)	(0.051)	(0.054)	(0.067)	(0.073)	(0.085)	(0.095)	(0.108)	(0.076)
Average y_t/Y_t	0.021	0.029	0.038	0.049	0.074	0.105	0.133	0.184	0.249	0.380	0.733
	(0.035)	(0.041)	(0.046)	(0.053)	(0.063)	(0.073)	(0.080)	(0.094)	(0.097)	(0.111)	(0.073)

Note: The table reports the share of elections in which the absolute difference in spending ratios is less than 0.1 and 0.05 for every week, across different election types. We define close elections to be races where the final difference in vote shares between two candidates is less than 5 percentage points. We define races in which the budgets are close to be races where the ratio of budgets of the two candidates are in the interval $(0.75, 1.25)$.

races than in House races, and violated to a greater extent as election day approaches. That said, the absolute difference in spending ratios is less than 0.1 for 85% of our dataset, and less than 0.05 for 65%. Even in the final six weeks where all candidates spend a positive amount, the candidates' spending ratios are within 10 percentage points of one another in 75.4% of election-weeks, and within 5 percentage points of one another in 49.5% of them. So, while there is a substantial amount of violation of the equal spending ratio result, the extent of violations seem to be limited.

In addition to looking at Senate, gubernatorial and House races separately, we also look in Table 2 separately at (i) elections with early voting versus those without, (ii) those that are open seat versus those in which an incumbent is running, (iii) those in which the final vote difference between the top two candidates is less than 5 percentage points versus those with larger margins, and (iv) those in which one candidate's budget is more than 25 percent greater than the other's, versus those where it is not. We do not find major differences in the extent to which the equal spending ratio result is violated across these settings apart from the observation that it appears to be violated less in close elections and in those with more symmetric budgets.

6 Conclusion

We have proposed a new model of dynamic campaigning, and used it to recover estimates of the decay rate in the popularity process using spending data alone.

Our theoretical contribution raises new questions, however. Since we focused on the strategic choices made by the campaigns, we abstracted away from some important considerations. For example, we left unmodeled the behavior of the voters that generates over-time fluctuations in relative popularity. In addition, we abstracted away from the motivations and choices of the donors, and the effort decisions of the candidates in how much time to allocate to campaigning versus fundraising. These abstractions leave open questions about how to micro-found the behavior of voters and donors, and effort allocation decision for the candidates.[25]

final week, each candidate spends 100% of money left over, so if we added the final (partial) twelfth week of the election, to the final column, these numbers would all be 100%, by construction.

[25]Bouton et al. (2018) address some of these questions in a static model. They study the strategic choices of donors who try to affect the electoral outcome and show that donor behavior depends on

Moreover, we have abstracted from the fact that in real life, campaigns may not know what the return to spending is at various stages of the campaign, or what the decay rate is, as these may be specific to the personal characteristics of the candidates, and changes in the political environment, including the "mood" of voters. Real-life campaigns face an optimal experimentation problem whereby they try to learn about their environment through early spending. Our model also abstracted away from the question of how early spending may benefit campaigns by providing them with information about what kinds of campaign strategies seem to work well for their candidate. This is a considerably difficult problem, especially in the face of a fixed election deadline, and the endogeneity of donor interest and available resources. But there is no doubt that well-run campaigns spend to acquire valuable information about how voters are engaging with and responding to the candidates over the course of the campaign. These are interesting and important questions that ought to be addressed in subsequent work.

the competitiveness of the election. Similarly, Mattozzi and Michelucci (2017) analyze a two-period dynamic model in which donors decide how much to contribute to each of two possible candidates without knowing ex-ante who is the more likely winner.

Appendix

A Proofs

A.1 Proof of Theorem 1

Existence of an interior equilibrium under the conditions posited in the proposition, and independence of spending ratios from the history of relative popularity, both follow from the argument in the main text above the theorem.

To prove that Assumption A implies the equal spending ratio result, write Z_T as in equation (3) in the main text, and note that at any time $t \in \mathcal{T}$ candidate 1 maximizes $\Pr\left[Z_T \geq 0 \mid z_t, X_t, Y_t\right]$ under the constraint $\sum_{n=t/\Delta}^{N-1} x_{n\Delta} \leq X_t$, while candidate 2 minimizes this probability under the constraint $\sum_{n=t/\Delta}^{N-1} y_{n\Delta} \leq Y_t$.

Consider the final period. Because money-left over has no value, candidates will spend all of their remaining budget in the last period so that the equal spending ratio result holds trivially in the last period.

Now consider any period m that is not the final period. Candidate 1 will maximize the mean of Z_T while candidate 2 minimizes it. By the budget constraint, this implies that equilibrium spending $x_{n\Delta}$ and $y_{n\Delta}$ for any period $n \in \{0, 1, ..., N-2\}$ solve the following pair of first order conditions

$$e^{-\lambda\Delta(N-1-n)} p_x(x_{n\Delta}, y_{n\Delta}) = p_x\left(X_0 - \sum_{m=0}^{N-2} x_{m\Delta}, Y_0 - \sum_{m=0}^{N-2} y_{m\Delta}\right)$$
$$e^{-\lambda\Delta(N-1-n)} p_y(x_{n\Delta}, y_{n\Delta}) = p_y\left(X_0 - \sum_{m=0}^{N-2} x_{m\Delta}, Y_0 - \sum_{m=0}^{N-2} y_{m\Delta}\right)$$

Taking the ratio of these first order conditions, applying Assumption A and inverting function ψ, we get that $\forall n < N-2$

$$\frac{x_{n\Delta}}{X_0 - \sum_{m=0}^{N-2} x_{m\Delta}} = \frac{y_{n\Delta}}{Y_0 - \sum_{m=0}^{N-2} y_{m\Delta}}.$$

or equivalently

$$x_{n\Delta} = \frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}}\, y_{n\Delta}$$

Thus for every $n < N - 2$, we have

$$\frac{x_{n\Delta}}{X_{n\Delta}} = \frac{x_{n\Delta}}{\sum_{m=n}^{N-1} x_{m\Delta}} = \frac{\frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}} y_{n\Delta}}{\sum_{m=n}^{N-2} \left(\frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}} y_{m\Delta} \right) + x_{(N-1)\Delta}} = \frac{y_{n\Delta}}{\sum_{m=n}^{N-1} y_{m\Delta}} = \frac{y_{n\Delta}}{Y_{n\Delta}}.$$

Therefore, the equal spending result holds for all periods.

A.2 Details for Remark 4

Proposition A.1. *In any equilibrium of the multi-district extension described in Remark 4, if $X_t, Y_t > 0$ are the remaining budgets of candidates 1 and 2 at any time $t \in \mathcal{T}$, then for all districts s,*

$$x_t^s / X_t = y_t^s / Y_t.$$

Proof. Note that the game ends in a defeat for any candidate that spends 0 in any district in any period. Therefore, in equilibrium spending must be interior (i.e., satisfy the first order conditions) for any district and any period.

Given this, we will prove the proposition by induction. Consider the final period as the basis case. Fix $\left(z_{T-\Delta}^s\right)_{s=1}^S$ arbitrarily. Suppose candidates 1 and 2 have budgets X and Y, respectively in the last period. Fix an equilibrium strategy profile $(x_{T-\Delta}^{s,*}, y_{T-\Delta}^{s,*})_{s=1}^S$. We show that, if they have budgets θX and θY, then $(\theta x_{T-\Delta}^{s,*}, \theta y_{T-\Delta}^{s,*})_{s=1}^S$ is an equilibrium. This implies that the equilibrium payoff in the last period is determined by $\left(z_{T-\Delta}^s\right)_{s=1}^S$ and $X_{t-\Delta}/Y_{t-\Delta}$.

Suppose otherwise. Without loss, assume that there is $\left(\tilde{x}_{T-\Delta}^{s,*}\right)_{s=1}^S$ such that it gives a higher probability of winning to candidate 1 given $\left(z_{T-\Delta}^s\right)_{s=1}^S$ and $\theta y_{T-\Delta}^{s,*}$, satisfying $\sum_{s=1}^S \tilde{x}_{T-\Delta}^{s,*} \leq \theta X$. Since the distribution of $(Z_T^s)_{s=1}^S$ is determined by $\left(z_{T-\Delta}^s\right)_{s=1}^S$ and $\left(x_{t-\Delta}^s / y_{t-\Delta}^s\right)_{s=1}^S$, this means that the distribution of $(Z_T^s)_{s=1}^S$ given $\left(z_{T-\Delta}^s\right)_{s=1}^S$ and $\left(\tilde{x}_{t-\Delta}^s / \theta y_{t-\Delta}^{*,s}\right)_{s=1}^S$ is more favorable to candidate 1 than that given $\left(z_{T-\Delta}^s\right)_{s=1}^S$ and

$$\left(\theta x_{t-\Delta}^{*,s} / \theta y_{t-\Delta}^{*,s}\right)_{s=1}^S = \left(x_{t-\Delta}^{*,s} / y_{t-\Delta}^{*,s}\right)_{s=1}^S.$$

On the other hand, candidate 1 could spend $\left(\frac{1}{\theta} \tilde{x}_{T-\Delta}^{s,*}\right)_{s=1}^S$ when the budgets are (X, Y). Since $(x_{T-\Delta}^{s,*}, y_{T-\Delta}^{s,*})_{s=1}^S$ is an equilibrium, the distribution of $(Z_T^s)_{s=1}^S$ given $(z_{T-\Delta}^s)_{s=1}^S$ and

$\left(\frac{1}{\theta}\tilde{x}^s_{t-\Delta}/y^{*,s}_{t-\Delta}\right)^S_{s=1} = \left(\tilde{x}^s_{t-\Delta}/\theta y^{*,s}_{t-\Delta}\right)^S_{s=1}$ is no more favorable to candidate 1 than that given $(z^s_{T-\Delta})^S_{s=1}$ and $(x^{*,s}_{t-\Delta}/y^{*,s}_{t-\Delta})^S_{s=1}$. This is a contradiction.

Now consider the inductive step. Take the inductive hypothesis to be that the continuation payoff for either candidate in period $t \in \mathcal{T}$ can be written as a function of only the budget ratio X_{t+1}/Y_{t+1} and vector $\left(z^s_{t+1}\right)^S_{s=1}$ and candidates spend a positive amount in each district and in each following period. We have to show that $x^s_t/X_t = y^s_t/Y_t$.

For all τ, let $x_\tau := \sum_s x^s_\tau$, $y_\tau := \sum_s y^s_\tau$ and $z_\tau := (z^s_\tau)^S_{s=1}$. Let $V_{t+1}\left(\frac{X_{t+1}}{Y_{t+1}}, z_{t+1}\right)$ denote the continuation payoff of candidate 1 starting in period $t+1$.

Candidate 1's objective is

$$\max_{(x^s_t)^S_{s=1}} \int V_{t+1}\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1}\right) f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right) dz_{t+1}.$$

where $f_t(\cdot|\cdot)$ denote the conditional distribution of the vector z_{t+1}. The first order conditions for an interior optimum for candidate 1 are then: for all $s \in \{1, ..., S\}$,

$$\frac{1}{Y_t - y_t} \int \frac{\partial V_{t+1}\left((X_t - x_t)/(Y_t - y_t), z_{t+1}\right)}{\partial (x^s_t/y^s_t)} f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right) dz_{t+1} =$$
$$= \frac{1}{y^s_t} \int V_{t+1}\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1}\right) \frac{\partial f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right)}{\partial (x^s_t/y^s_t)} dz_{t+1}.$$

Similarly, the objective for candidate 2 is

$$\min_{(y^s_t)^S_{s=1}} \int V_{t+1}\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1}\right) f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right) dz_{t+1}.$$

and the corresponding first order conditions are: for all $s \in \{1, ..., S\}$,

$$\frac{X_t - x_t}{\left(Y_t - y_t\right)^2} \int \frac{\partial V_{t+1}\left((X_t - x_t)/(Y_t - y_t), z_{t+1}\right)}{\partial (x^s_t/y^s_t)} f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right) dz_{t+1}$$
$$= \frac{x^s_t}{\left(y^s_t\right)^2} \int V_{t+1}\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1}\right) \frac{\partial f_t\left(z_{t+1} \mid \left(x^s_t/y^s_t\right)^S_{s=1}, z_t\right)}{\partial (x^s_t/y^s_t)} dz_{t+1}.$$

Dividing the candidate 1's first order condition by candidate 2's, we have

$$\frac{X_t - x_t}{Y_t - y_t} = \frac{x_t^s}{y_t^s}.$$

Hence there exists θ such that $x_t^s = \theta y_t^s$ for all s, and so

$$\theta = \frac{X_t - \theta y_t}{Y_t - x_t},$$

which implies $\theta = X_t/Y_t$. Therefore, $x_t^s/y_t^s = X_t/Y_t$ for all s. ■

A.3 Proof of Proposition 2

For the periods $\hat{N}$,..., N, we can write

$$\begin{aligned}
Z_{N\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta N} + \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}, \\
Z_{(N-1)\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{N-2} e^{-\lambda\Delta(N-2-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta(N-1)} + \sum_{n=0}^{N-2} e^{-\lambda\Delta(N-2-n)} \varepsilon_{n\Delta}, \\
&\vdots \\
Z_{\hat{N}\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta\left(\hat{N}-1-n\right)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta\hat{N}} + \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta\left(\hat{N}-1-n\right)} \varepsilon_{n\Delta}.
\end{aligned}$$

Substituting these in the objective function of the candidates, we can rewrite it as:

$$\Pr\left\{\sum_{m=0}^{N-\hat{N}} \xi^m Z_{(N-m)\Delta} \geq 0\right\} = \Pr\left\{\sum_{m=0}^{N-\hat{N}} \xi^m E_{N-m} \geq -\sum_{m=0}^{N-\hat{N}} \xi^m B_{N-m}\right\},$$

where

$$B_k := \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{k-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta k}$$

and

$$E_k := \sum_{n=0}^{k-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}$$

Because all E_k are sums of normally distributed shocks, we can equivalently assume that candidate 1 maximizes, and 2 minimizes $\sum_{m=0}^{N-\hat{N}} \xi^m B_{N-m}$. Hence, candidate 1 maximizes, and 2 minimizes:

$$\sum_{k=0}^{N-(\hat{N}+1)} \left(\sum_{m=0}^{k} \xi^m e^{-\lambda\Delta(k-m)} \right) p(x_{(N-1-k)\Delta}, y_{(N-1-k)\Delta}) + \\ + \left(\sum_{m=0}^{N-\hat{N}} \xi^m e^{-\lambda\Delta(k-m)} \right) \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta\left(\hat{N}-1-n\right)} p(x_{n\Delta}, y_{n\Delta})$$

and it is clear from this that under Assumption A the equal spending ratio holds.

Now, for any two consecutive periods both prior to period $\hat{N}$, after we cancel out the constant terms, the consecutive period spending ratio is the same as the one derived for the example in Section 3.2, hence it is constant. Consider two consecutive periods $(\hat{N}+k)$ and $(\hat{N}+k+1)$, with $k \in \{0, ..., N-\hat{N}-2\}$. Reasoning as in that example, if we equate the first order conditions for these two periods we get

$$x^{\beta}_{\hat{N}+k} h'(x_{\hat{N}+k}, y_{\hat{N}+k}) = e^{\lambda\Delta} \left(\frac{(e^{-\lambda\Delta}/\xi) - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}}{1-(e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}} \right) x^{\beta}_{\hat{N}+k+1} h'(x_{\hat{N}+k+1}, y_{\hat{N}+k+1}).$$

Because the term in parentheses above is lower than 1, if we compare this equation with equation (4), we can show that the consecutive period spending ratio is now lower. In particular, using the same steps used to prove Proposition 1, we get that the consecutive period spending ratio is

$$\hat{r}_{\hat{N}+k} = e^{-\frac{\lambda\Delta}{\beta}} \left(\frac{(e^{-\lambda\Delta}/\xi) - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}}{1-(e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}} \right)^{-\frac{1}{\beta}}.$$

and the term in parentheses is lower than 1. Observe that the term in parentheses is decreasing in k. Therefore, for $k \geq 0$, $\hat{r}_{\hat{N}+k}$ will be decreasing in k since $\beta < 0$.

For the same reason, the term in parenthesis is also increasing in $e^{-\lambda\Delta}/\xi$ and thus the consecutive period spending ratio is decreasing in ξ.

A.4 Proof of Proposition 3

We will in fact prove a more general result than Proposition 3 under which we also characterize the stochastic path of spending over time for this extension.

Applying Itô's lemma, we can write the process governing the evolution of this ratio for this model as:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_{XY}(z_t)dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y, \tag{5}$$

where

$$\mu_{XY}(z_t) = (a-b)z_t + \sigma_Y^2 - \rho\sigma_X\sigma_Y.$$

Hence, the instantaneous volatility of this process is simply $\sigma_{XY} = \sqrt{\sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y}$. Therefore, if at time $t \in \mathcal{T}$ the candidates have an amount of available resources equal to X_t and Y_t and spend x_t and y_t, then $Z_{t+\Delta}$ conditional on all information, $\mathcal{I}_t$, available at time time t is a normal random variable:

$$Z_{t+\Delta} \mid \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{x_t}{y_t}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\right),$$

and Itô's lemma implies that

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \mid \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + \mu_{XY}(z_t)\Delta, \sigma_{XY}^2\Delta\right).$$

Last, let $g_1(0) = 1$ and $g_2(0) = 0$, and define recursively for every $m \in \{1, ..., N-1\}$,

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} & a-b \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} g_1(m-1) \\ g_2(m-1) \end{pmatrix} \tag{6}$$

Then we have the following result, which implies Proposition 3 in the main text.

Proposition A.2. *Let $t = (N-m)\Delta \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in the essentially unique equilibrium, spending ratios are equal to*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{g_1(m-1)}{g_1(m-1) + g_2(m-1)\frac{\lambda}{1-e^{-\lambda\Delta}}}. \tag{7}$$

Moreover, in equilibrium, $(\log(x_{t+n\Delta}/y_{t+n\Delta}), z_{t+n\Delta}) \mid \mathcal{I}_t$ *follows a bivariate normal distribution with mean*

$$\begin{pmatrix} 1 & (a-b)\,\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \log\left(\frac{X_t}{Y_t}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} - \begin{pmatrix} \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{a-b} \end{pmatrix}$$

and variance

$$\begin{pmatrix} 1 & (a-b)\,\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \sigma_{XY}^2\Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-e^{-\lambda\Delta}}{\lambda} \\ (a-b)\,\Delta & e^{-\lambda\Delta} \end{pmatrix}^n.$$

Proof. Consider time $t = n\Delta \in \mathcal{T}$ and suppose that at time t both candidates have still a positive budget, $X_t, Y_t > 0$. We will prove the proposition by induction on the times at which candidates take actions, $t = (N-m)\Delta \in \mathcal{T}$, $m = 1, 2, ..., N$.

To simplify notation, let $g_1(0) = 1, g_2(0) = 0, g_3(0) = 0$ and $g_4(0) = 0$. Furthermore, using (6), recursively write for every $m \in \{1, 2, ..., N\}$,

$$g_3(m) = g_2(m-1)\Delta + g_3(m-1)$$
$$g_4(m) = (g_1(m-1))^2 \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} + (g_2(m-1))^2\sigma_{XY}^2\Delta + g_4(m-1)$$

Diagonalizing the matrix in (6) and solving for $(g_1(m), g_2(m))'$ with initial conditions $g_0(1) = 1$ and $g_2(0) = 0$, we can conclude that, for each $N \in \mathbb{N}$ and $\lambda, \Delta > 0$, there exists $-\eta < 0$ such that, if $a - b \geq -\eta$, both $g_1(m)$ and $g_2(m)$ are non-negative for each m. In the proof, we will thus assume that $g_1(m) \geq 0$ and $g_2(m) \geq 0$ for every $m = 1, ..., N$.

The inductive hypothesis is the following: for every $\tau = (N-m)\Delta \in \mathcal{T}$, $m \in \{1, ..., N\}$, if $X_\tau, Y_\tau > 0$, then

(i) the continuation payoff of each candidate is a function of current popularity z_τ, current budget ratio X_τ/Y_τ and calendar time τ;

(ii) the distribution of Z_T given z_τ and X_τ/Y_τ is $\mathcal{N}\left(\hat{\mu}_{(N-m)\Delta}(z_\tau), \hat{\sigma}^2_{(N-m)\Delta}\right)$, where

$$\hat{\mu}_{(N-m)\Delta}(z_{(N-m)\Delta}) = g_1(m) z_{(N-m)\Delta} + g_2(m)\log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m)(\sigma_Y^2 - \rho\sigma_X\sigma_Y),$$

$$\hat{\sigma}^2_{(N-m)\Delta} = g_4(m).$$

Base Step Consider $m = 1$, the subgame reached in the final period $t = (N-1)\Delta$ and suppose both candidates still have a positive amount of resources, $X_{(N-1)\Delta}, Y_{(N-1)\Delta} > 0$. Both candidates will spend their remaining resources: $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Hence, $x_{(N-1)\Delta}/y_{(N-1)\Delta} = X_{(N-1)\Delta}/Y_{(N-1)\Delta}$ and

$$Z_T \mid \mathcal{I}_{(N-1)\Delta} \sim \mathcal{N}\left(\log\left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta}e^{-\lambda\Delta}, \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\right).$$

Because Z_T fully determines the candidates' payoffs, the continuation payoff of the candidates is a function of current popularity $z_{(N-1)\Delta}$, the ratio $X_{(N-1)\Delta}/Y_{(N-1)\Delta}$, and calendar time. Furthermore, given the recursive definition of g_1, g_2, g_3 and g_4, we can conclude that the second part of the inductive hypothesis also holds at $t = (N-1)\Delta$. This concludes the base step.

Inductive Step Suppose the inductive hypothesis holds true at any time $(N-m)\Delta \in \mathcal{T}$ with $m \in \{1, 2, ..., m^* - 1\}$, $m^* \leq N$. We want to show that at time $(N - m^*)\Delta \in \mathcal{T}$, if $X_t, Y_t > 0$, then (i) an equilibrium exists, (ii) in all equilibria, $x_t/y_t = X_t/Y_t$ and the continuation payoffs of both candidates are functions of relative popularity z_t, the ratio X_t/Y_t, and calendar time t, and (iii) Z_T given period t information is distributed according to $\mathcal{N}\left(\hat{\mu}_{(N-m^*)\Delta}(z_t), \hat{\sigma}^2_{(N-m^*)\Delta}\right)$.

Consider period $t = N - m^*$ and let $x, y > 0$ be the candidates' spending in this period. Exploiting the inductive hypothesis, the distribution of $Z_{t+\Delta} \mid \mathcal{I}_t$ and the one of $\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \mid \mathcal{I}_t$, we can compound normal distributions and conclude that $Z_T \mid \mathcal{I}_t \sim$

$\mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$, where

$$\begin{aligned}\tilde{\mu} &= \hat{\mu}_t(x,y) := G_1 \log\left(\frac{x}{y}\right) + G_2 \log\left(\frac{X_{(N-m^*)\Delta} - x}{Y_{(N-m^*)\Delta} - y}\right) + G_3 \\ \tilde{\sigma}^2 &= G_4\end{aligned}$$

with G_1, G_2, G_3 and G_4 defined as follows:

$$G_1 = g_1(m^* - 1)\frac{1 - e^{-\lambda\Delta}}{\lambda} \tag{8}$$

$$G_2 = g_2(m^* - 1) \tag{9}$$

$$G_3 = g_1(m^* - 1)z_t e^{-\lambda\Delta} + g_2(m^* - 1)\mu_{XY}(z_t)\Delta + g_3(m^* - 1)(\sigma_Y^2 - \rho\sigma_X\sigma_Y) \tag{10}$$

$$G_4 = (g_1(m^* - 1))^2 \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda} + (g_2(m^* - 1))^2\sigma_{XY}^2\Delta + g_4(m^* - 1) \tag{11}$$

Note that $\tilde{\sigma}^2$ is independent of x and y.

Candidate 1 wins the election if $Z_T > 0$. Thus, in equilibrium he chooses x to maximize his winning probability

$$\int_{\frac{-\hat{\mu}_t(x,y)}{\tilde{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s/2} ds.$$

The first order necessary condition for x is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\hat{\sigma}_t}} \frac{\hat{\mu}_t'(x,y)}{\tilde{\sigma}} = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x) - G_2 x}{x(X_t - x)}\right].$$

Furthermore, when the first order necessary condition holds, the second order condition is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\hat{\sigma}_t}} \frac{\hat{\mu''}_t(x,y)}{\tilde{\sigma}} = \frac{-1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x)^2 + G_2 x^2}{x^2(X_t - x)^2}\right] < 0.$$

Hence, the problem is strictly quasi-concave for candidate 1 for each y. A symmetric argument shows that the corresponding problem for candidate 2 is strictly quasi-concave for each x. Hence an equilibrium exists and the optimal investment of the two candidates is pinned down by the first order necessary conditions, which yields

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{G_1}{G_1 + G_2}. \tag{12}$$

Thus, in equilibrium, $x_t/y_t = X_t/Y_t$ and $(X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Because the continuation payoffs of candidates is fully determined by Z_T, these expected payoffs from the perspective of time t depend only on calendar time, the level of current popularity and the ratio of budget at time t. Furthermore, recalling the definition of $\mu_{XY}(z_t)$, we conclude that the second part of the inductive hypothesis is also true.

Next, we know that

$$Z_T \mid \mathcal{I}_{(N-m^*)\Delta} \sim \mathcal{N}(\hat{\mu}_{(N-m^*)\Delta}, \hat{\sigma}^2_{(N-m^*)\Delta})$$

where

$$\hat{\mu}_{(N-m^*)\Delta}(z_{(N-m^*)\Delta}) = g_1(m^*) z_{(N-m^*)\Delta} + g_2(m^*) \log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m^*)(\sigma_Y^2 - \rho\sigma_X\sigma_Y),$$

$$\hat{\sigma}^2_{(N-m^*)\Delta} = g_4(m^*).$$

The expression for x_t/X_t and y_t/Y_t in the proposition thus follows from (6), (8), (9) and (12). To derive the distribution of $(x_t/y_t, z_t)$, we first use the proof of Proposition 3 to derive the distribution of $x_{t+j\Delta}/y_{t+j\Delta}$ and $z_{t+j\Delta}$ given x_t/y_t and z_t. Let

$$\Sigma = \begin{pmatrix} \sigma_{XY}^2 \Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix}.$$

Because $X_t/Y_t = x_t/y_t$ for each t, we can write

$$\left(\begin{array}{c} \log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right) \\ z_{t+n\Delta} \end{array}\right) \Bigg| \left(\begin{array}{c} \frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \\ z_{t+(n-1)\Delta} \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right) + \mu_{XY}(z_{t+(n-1)\Delta})\Delta \\ \log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right) \frac{1-e^{-\lambda\Delta}}{\lambda} + z_{t+(n-1)\Delta} e^{-\lambda\Delta} \end{array}\right), \Sigma\right)$$

Define

$$A = \begin{pmatrix} 1 & (a-b)\,\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}.$$

and notice that the previous distribution implies

$$\left(\begin{array}{c} \log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \Bigg| \left(\begin{array}{c} \log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right)$$

follows a multivariate normal distribution

$$\mathcal{N}\left(A \left(\begin{array}{c} \log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right), \Sigma \right)$$

Therefore, we conclude that

$$\left(\begin{array}{c} \log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \Bigg| \left(\begin{array}{c} \log\left(\frac{x_t}{y_t}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right)$$

follows the multivariate normal distribution

$$\mathcal{N}\left(A^n \left(\begin{array}{c} \log\left(\frac{X_t}{Y_t}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right), A^n\Sigma(A^T)^n \right).$$

■

A.5 Proof of Proposition 4

Fix λ and Δ. and let $n = N - m$. We must show that for all $n \in \{0, ..., N-1\}$,

$$\check{r}_n(a-b) = \frac{x_{n\Delta}}{X_{n\Delta}} / \frac{x_{(n+1)\Delta}}{X_{(n+1)\Delta}}$$

is decreasing in $\alpha := a - b$ around $\alpha = 0$. Note that $\check{r}_n$ is the same as $\tilde{r}_{N-m}$.

Proposition A.2 and (6) imply

$$\check{r}_m(\alpha) = \frac{g_1(m-1)\left(g_1(m) + g_2(m)\frac{\lambda}{1-e^{-\lambda\Delta}}\right)}{\left(g_1(m-1) + g_2(m-1)\frac{\lambda}{1-e^{-\lambda\Delta}}\right)g_1(m)} = \frac{g_1(m-1)}{g_1(m)}\frac{g_2(m+1)}{g_2(m)}.$$

Furthermore, (6) also implies

$$g_1(m) = \frac{(\lambda+\alpha)e^{-\lambda\Delta} - \alpha}{\lambda} g_1(m-1) + \alpha g_2(m), \tag{13}$$

$$g_2(m+1) = \frac{\left(1-e^{-\lambda\Delta}\right)\left((\lambda+\alpha)e^{-\lambda\Delta} - \alpha\right)}{\lambda^2} g_1(m-1) + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} g_2(m). \tag{14}$$

Substituting in the expression for $\check{r}_m(\alpha)$ and simplifying, we get

$$\check{r}_m(\alpha) = \frac{1}{\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \alpha g_m}\left(\frac{\left(1-e^{-\lambda\Delta}\right)\left((\lambda+\alpha)e^{-\lambda\Delta} - \alpha\right)}{\lambda^2}\frac{1}{g_m} + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda}\right) \tag{15}$$

where $g_m := g_2(m)/g_1(m-1)$. We can thus identify two values of g_m for which (15) holds. However, if α is sufficiently low, namely if $\alpha < \lambda/(1+e^{\lambda\Delta})$, one of these two values is negative and thus not feasible. Thus, if α is sufficiently small, (15) enables us to express g_m as a function of $\check{r}_m(\alpha)$. Moreover, from (13) and (14), we further have

$$g_{m+1} = \frac{\frac{1-e^{-\lambda\Delta}}{\lambda}\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \frac{\alpha+\lambda-\alpha e^{-\lambda\Delta}}{\lambda}g_m}{\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \alpha g_m}. \tag{16}$$

Computing (15) one step forward and substituting for g_{m+1} as obtained from (16) and, subsequently, for g_m as obtained from (15), we get $\check{r}_{m+1}$ as a function of α and $\check{r}_m$, written $\check{r}_{m+1}(\alpha, \check{r}_m)$.

Given the expression for $\check{r}_{m+1}$, we can show by induction that $\check{r}_m > e^{\lambda\Delta} > 1$ for each m around $\alpha = 0$. When $m = 1$, we have $x_{(N-1)\Delta}/X_{(N-1)\Delta} = 1$ and $x_{(N-2)\Delta}/X_{(N-2)\Delta} = g_1(1)/\left(g_1(1) + g_2(1)\frac{\lambda}{1-e^{-\lambda\Delta}}\right)$. Substituting for $g_1(1)$ and $g_2(1)$, we get $\check{r}_1 - e^{\lambda\Delta} = 1$. Thus, $\check{r}_1 > e^{\lambda\Delta} > 1$. Suppose $\check{r}_m > e^{\lambda\Delta} > 1$. Then, subtracting $e^{\lambda\Delta}$ from the right hand side of the expression of $\check{r}_{m+1}$ and setting $\alpha = 0$, we get $\check{r}_{m+1} - e^{\lambda\Delta} = 1 - e^{\lambda\Delta}/\check{r}_m > 0$.

We conclude that, if $\check{r}_m > e^{\lambda\Delta}$, then $\check{r}_{m+1} > e^{\lambda\Delta}$ in a neighborhood of $\alpha = 0$. Therefore, $\check{r}_m > e^{\lambda\Delta}$ for each m in a neighborhood of $\alpha = 0$.

Furthermore, $\check{r}_{m+1}(\alpha, \check{r}_m)$ is decreasing in α and increasing in $\check{r}_m$ at $\alpha = 0$:

$$\left.\frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m)}{\partial \alpha}\right|_{\alpha=0} = -\frac{(\check{r}_m - 1)\, e^{\lambda\Delta}\left(e^{2\lambda\Delta} - 1\right)}{\check{r}_m\left(\check{r}_m - e^{\lambda\Delta}\right)} < 0;$$
$$\left.\frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m)}{\partial \check{r}_m}\right|_{\alpha=0} = \frac{e^{\lambda\Delta}}{(\check{r}_m)^2} > 0.$$

Hence, a simple induction argument implies that $\check{r}_m(\alpha)$ is decreasing in α for each m in a neighborhood of $\alpha = 0$.

Finally, $\check{r}_m$ is increasing in λ as well:

$$\left.\frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m, \lambda)}{\partial \lambda}\right|_{\alpha=0} = \frac{e^{\lambda\Delta}(\check{r}_m - 1)\,\Delta}{\check{r}_m} > 0 \text{ for each } \lambda > 0.$$

Thus, a symmetric inductive argument shows that $\check{r}_m$ is increasing in λ for every m in a neighborhood of $\alpha = 0$.

A.6 Details for Remark 5

Proposition A.3. *In the model with endogenous budgets that evolve depending on the closeness of the race, if for all $t \in \mathcal{T}$, $X_t, Y_t > 0$, then in equilibrium,*

$$x_t/X_t = y_t/Y_t.$$

Proof. For any $t \in \mathcal{T}$ the distribution of $Z_{t+\Delta} \mid \mathcal{I}_t$ is given by (2), while Ito's lemma implies:

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \mid \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + m(z_t), \sigma_{XY}^2\Delta\right), \tag{17}$$

where $m_{XY}(z_t) = (a-b)/(1+z_t^2) + \sigma_Y^2 - \rho\sigma_X\sigma_Y$ and $\sigma_{XY}^2 = \sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y$. Furthermore, the two distributions are independent (conditional on $\mathcal{I}_t$). Let ϕ_1 and ϕ_2 be the pdfs of these two distributions. The proof is by induction.

Base Step. Consider period $t = (N-1)\Delta$. Because money leftover has no value and we are considering an interior equilibrium, $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Thus, the equal spending holds at time $t = (N-1)\Delta$. Also, observe that the continuation payoff of candidates is fully determined by the distribution of Z_T and, in equilibrium,

$$Z_T \mid \mathcal{I}_{(N-1)\Delta} \sim \mathcal{N}\left(\log\left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta}e^{-\lambda\Delta}, \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\right).$$

Hence, in equilibrium, the expected continuation payoff of candidates at time $(N-1)\Delta$ depends on the popularity at time $(N-1)\Delta$, $z_{(N-1)\Delta}$, and on the logarithm of the available budgets, $\log(X_{(N-1)\Delta}/Y_{(N-1)\Delta})$. Denote such an expected continuation payoff for candidate 1 with $V_{(N-1)\Delta}(z_{(N-1)\Delta}, X_{(N-1)\Delta}/Y_{(N-1)\Delta})$. Obviously, the expected continuation payoff for candidate 2 is $1 - V_{(N-1)\Delta}(z_{(N-1)\Delta}, X_{(N-1)\Delta}/Y_{(N-1)\Delta})$.

Inductive Step. Pick $m \in \{0, N-2\}$ and suppose that for all periods $\tau \in \{(N-m+1)\Delta, (N-m+2)\Delta, (N-1)\Delta\}$ in an interior equilibrium the equal spending ratio result holds and the expected continuation payoff of candidates depends on z_τ, and on X_τ and Y_τ only through the log of their ratio, $\log(X_\tau/Y_\tau)$. Denote this continuation for candidate 1 with $V_\tau(z_\tau, X_\tau/Y_\tau)$. Then, at time $t = (N-m)\Delta$, the expected payoff of candidate 1 is:

$$\begin{aligned} V_t(z_t, x_t, y_t) = \int \phi_1(z_{t+\Delta} \mid z_t, x_t, y_t)\phi_2\left(z_{t+\Delta}, \log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \mid z_t, x_t, y_t\right)\times \\ \times V\left(z_{t+\Delta}, \log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right)\right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}). \end{aligned} \tag{18}$$

Candidate 1 chooses x_t to maximize $V_t(z_t, x_t, y_t)$ and candidate 2 chooses y_t to minimize it. Hence the two first order conditions are given by:

$$\frac{1}{x_t}\int \frac{\partial \phi_1}{\partial \mu_1}\phi_2 V_{t+\Delta} d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}) =$$
$$+\frac{1}{X_t - x_t}\int \phi_1 \left(\frac{\partial \phi_2}{\partial \mu_2} V_{t+\Delta} + \phi_2 \frac{\partial V_{t+\Delta}}{\partial \log\left(\frac{X_t - x_t}{Y_t - y_t}\right)}\right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta})$$
$$\frac{1}{y_t}\int \frac{\partial \phi_1}{\partial \mu_1}\phi_2 V_{t+\Delta} d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}) =$$
$$+\frac{1}{Y_t - y_t}\int \phi_1 \left(\frac{\partial \phi_2}{\partial \mu_2} V_{t+\Delta} + \phi_2 \frac{\partial V_{t+\Delta}}{\partial \log\left(\frac{X_t - x_t}{Y_t - y_t}\right)}\right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta})$$

Consider candidate 1 (the reasoning for candidate 2 is identical). Spending 0 at time t is not compatible with equilibrium behavior: if candidate 1 spends 0 at time t, a deviation to spending $X_\tau/(N-\tau)$ in all periods $\tau \geq t$ would strictly increase the winning probability.[26]. Hence both candidates must be spending a positive amount in period t. Similarly, $x_t = X_t$ cannot be compatible with equilibrium behavior either: if candidate 2 is spending a positive amount in period $(N-m+1)\Delta$, this strategy would lead to the defeat of player 1 in period $(N-m+1)\Delta$, while by spending $x_\tau = X_\tau/(N-\tau)$ for all $\tau \geq t$ candidate 1 could win with positive probability. (If candidate 2 is spending 0 in period $(N-m+1)\Delta$, $x_t = X_t$ would lead player 1 to win with probability 1/2, while $x_\tau = X_\tau/(N-\tau)$ for all $\tau \geq t$ would guarantee victory with probability 1.) Hence the equilibrium must be interior and the first order condition must hold. Thus, taking the ratio of the two first order conditions, we get the equal spending ratio result. Hence, in an interior equilibrium $x_t/y_t = (X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Furthermore, the expected continuation payoff in an interior equilibrium depends on the popularity z_t and on the initial budgets X_t, Y_t only through $\log(X_t/Y_t)$. ■

[26]The probability would jump from 0 to a positive amount if candidate 2 was spending a positive amount and from 1/2 to 1 if candidate 2 was spending 0

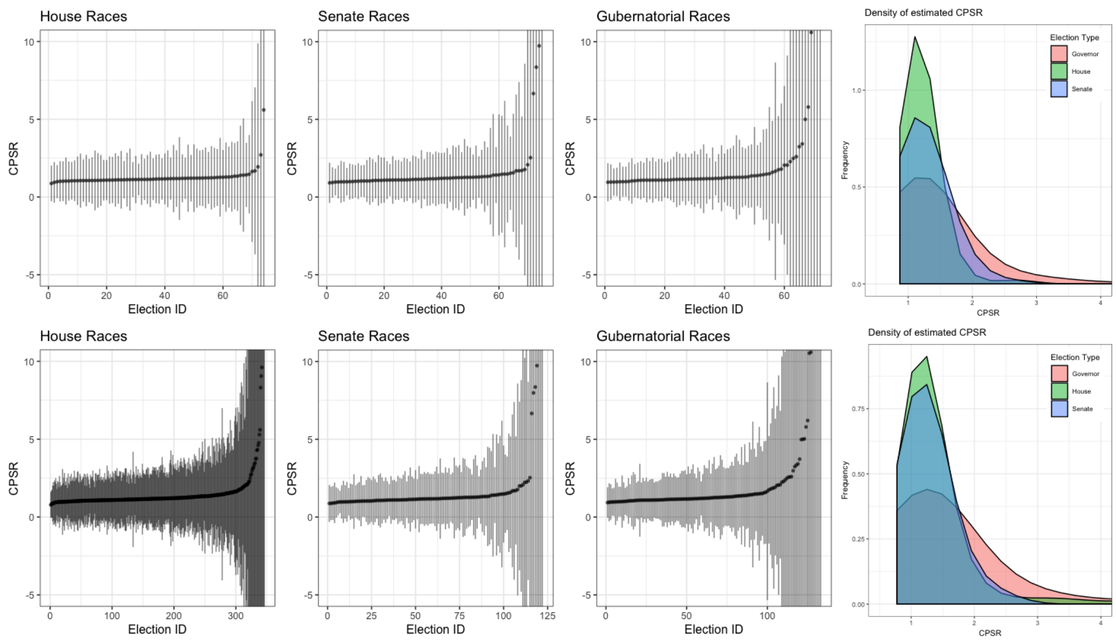

Figure B.1: Estimated CPSR values with 95% confidence intervals. The upper row are estimates of the CPSR that we get from dropping all elections with zero spending. The bottom row are estimates that we get from dropping all pairs of consecutive weeks that include zero spending. We also depict the densities of the CPSR across election types using both approaches.

B Examining the Constant Spending Growth Result

In this appendix, we investigate the extent to which the constant spending growth result of the example in Section 3.2 holds up in the data.

The consecutive period spending ratio (CPSR) is defined as x_{t+1}/x_t for the Democrat and y_{t+1}/y_t for the Republican candidate, over all twelve weeks t. If the equal spending ratio result holds, then these are the same for the two candidates. However, since there are candidates who spend zero in some weeks, this ratio cannot be defined for certain weeks. To deal with this problem, we take three different approaches.

We first calculate the consecutive period spending ratios for every candidate in the dataset using two approaches: (i) dropping all elections with zero spending in any week, and (ii) dropping all pairs of consecutive weeks that would include a week with zero spending.[27] These constitute two different rules for discarding data. Approach (i) leaves us with only 221 (out of the total 601) elections in our dataset where no zero spending occurs, and in approach (ii) we drop 1,692 consecutive week pairs out of a total of 13,223, which is only 12.8% of consecutive week pairs. Moreover, there is no instance of zero spending following positive spending in the sample: once a candidate starts spending a positive amount they continue to do so until the election.

The distribution of CPSRs along with their 95% confidence intervals from each of the two approaches are depicted in Figure B.1. The figure shows that while there are some important differences, the distributions are similar. Overall, the standard errors are typically higher using approach (ii), so to be conservative in assessing the extent of violations of the constant spending growth predictions, we proceed with the estimates from this approach.

Table B.1 shows that the constant spending growth prediction is violated to a smaller extent as the election approaches and candidates begin to spend more substantial amounts. The same table also shows that the statewide races, which typically see larger amounts of money spent, generally have smaller/fewer violations than House races. For example, even in the last eight weeks of the elections, the consecutive period spending ratios remain within 20% of their means for each candidate in 37.1%, 34.5%, and 25.9% of Senate, gubernatorial and House candidates, respectively.

One possible explanation for these violations is that our constant spending growth result does not hold in elections where there is early voting starting from the time that ballots can be cast. However, Table B.1 shows that looking only at races in which early voting is not allowed does not seem to reduce the extent of violations by much, though there is some improvement given that early voting typically starts two to seven weeks before election day depending on the state. Another possible explanation is that the result relies on the assumption that the candidates can correctly forecast how much money the will end up raising by the end of the campaign—which is not true in the evolving budgets extension—and it is hard for candidates to do this, especially for House candidates for whom the amount of money they will raise is more uncertain. Unfortunately, however, we cannot investigate whether the equilibrium spending path predicted by our evolv-

[27]If zero spending occurs at time t, both x_{t+1}/x_t and x_t/x_{t-1} are excluded.

Table B.1: Consecutive Period Spending Ratios

	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	Overall
% $\in$ (-0.5σ,0.5σ)	0.390	0.332	0.354	0.413	0.440	0.487	0.483	0.556	0.556	0.572	0.473
Senate	0.410	0.330	0.369	0.406	0.442	0.537	0.553	0.652	0.586	0.660	0.503
Governor	0.417	0.341	0.358	0.444	0.453	0.549	0.541	0.624	0.620	0.579	0.501
House	0.354	0.327	0.344	0.402	0.435	0.445	0.436	0.496	0.520	0.539	0.448
Early Voting	0.408	0.353	0.369	0.425	0.454	0.484	0.484	0.543	0.555	0.555	0.475
No Early Voting	0.357	0.291	0.323	0.391	0.414	0.493	0.483	0.581	0.556	0.605	0.468
Open Seat	0.390	0.328	0.344	0.388	0.448	0.525	0.502	0.565	0.578	0.625	0.482
Incumbent Competing	0.390	0.335	0.361	0.430	0.436	0.464	0.472	0.550	0.542	0.541	0.467
Close Election	0.335	0.299	0.344	0.430	0.444	0.515	0.524	0.550	0.526	0.579	0.470
Not Close Election	0.415	0.347	0.358	0.407	0.439	0.476	0.468	0.558	0.567	0.570	0.474
Close Budgets	0.414	0.315	0.372	0.412	0.446	0.506	0.500	0.538	0.600	0.602	0.483
Not Close Budgets	0.368	0.348	0.337	0.415	0.436	0.472	0.470	0.570	0.521	0.549	0.464
% $\in$ (0.8μ,1.2μ)	0.146	0.139	0.187	0.242	0.258	0.302	0.296	0.282	0.342	0.327	0.229
Senate	0.250	0.180	0.230	0.303	0.311	0.352	0.352	0.381	0.418	0.414	0.290
Governor	0.180	0.173	0.180	0.274	0.252	0.353	0.293	0.331	0.447	0.398	0.262
House	0.095	0.111	0.175	0.208	0.241	0.264	0.277	0.228	0.275	0.269	0.195
Early Voting	0.278	0.248	0.268	0.321	0.332	0.366	0.350	0.402	0.436	0.419	0.353
No Early Voting	0.291	0.227	0.274	0.306	0.331	0.407	0.390	0.453	0.446	0.456	0.372
Open Seat	0.277	0.228	0.270	0.314	0.318	0.408	0.364	0.420	0.442	0.444	0.358
Incumbent Competing	0.287	0.251	0.269	0.317	0.339	0.363	0.363	0.419	0.438	0.424	0.360
Close Election	0.288	0.208	0.294	0.348	0.343	0.397	0.391	0.418	0.426	0.444	0.366
Not Close Election	0.280	0.257	0.259	0.302	0.327	0.374	0.353	0.420	0.444	0.427	0.357
Close Budgets	0.307	0.245	0.314	0.336	0.361	0.423	0.397	0.444	0.506	0.468	0.391
Not Close Budgets	0.260	0.237	0.230	0.298	0.307	0.346	0.337	0.400	0.387	0.403	0.333

Note: The upper panel of the table reports the share of candidates for which the CPSRs are less than 0.5 standard deviations away from that candidate's average CPSR over 11 weeks. The lower panel reports the share of candidates in that week for which their CPSRs are within 20% of their average CPSR. The overall share is the share of candidate-weeks that fall within 20% of the corresponding candidate's average CPSR over all weeks. Week -2 is missing because the final week is not included in the analysis. See the note under Table 2 for the definition of close elections and close budgets.

ing budget extension could account for these violations since data on when candidates receive money or pledges from donors are not available.

Finally, we also look at the extent of violations of the constant spending growth prediction in the other disaggregations that we looked at with the equal spending ratio result in Section 5.2. Again we find overall small differences across the different settings, though the prediction is violated substantially more in elections where the budgets are asymmetric than those in which they are relatively close: the consecutive period spending ratios remain within 20% of their means for each candidate in 39.1% of races with close budgets, and only 33.3% of races with highly unequal budgets.

Senate Elections in our Sample

Year	State
2000	DE, FL, IN, ME, MI, MN, MO, NE, NV, NY, PA, RI, VA, WA
2002	AL, AR, CO, GA, IA, LA, ME, NC, NH, NJ, OK, OR, SC, TN, TX
2004	CO, FL, GA, KY, LA, NC, OK, PA, SC, WA
2006	AZ, MD, MI, MO, NE, OH, PA, RI, TN, VA, WA, WV
2008	AK, CO, GA, ID, KS, KY, LA, ME, MS, NC, NE, NH, NM, OK, OR, SD
2010	AL, AR, CA, CO, CT, IA, IL, IN, KY, LA, MD, MO, NH, NV, NY, OR, PA, VT, WA
2012	AZ, CT, FL, HI, IN, MA, MO, MT, ND, NE, NM, NV, OH, PA, RI, VA, WI, WV
2014	AK, AR, CO, GA, IA, IL, KY, LA, ME, MI, MT, NC, NH, NM, OR, SD, VA, WV

Gubernatorial Elections in our Sample

Year	State
2000	IN, MO, NC, NH, WA, WV
2002	AL, AR, AZ, CA, CT, FL, GA, HI, IA, IL, KS, MA, MD, ME, MI, NM, NY, OK, OR, PA, RI, SC, TN, TX, WI
2004	IN, MO, NC, NH, UT, VT, WA
2006	AL, AR, AZ, CO, CT, FL, GA, IA, IL, KS, MD, ME, MI, MN, NH, NV, NY, OH, OR, PA, RI, TN, VT, WI
2008	IN, MO, NC, WA
2010	AK, AL, AR, AZ, CA, CT, FL, GA, HI, IA, ID, IL, MA, MD, MI, MN, NH, NM, NV, NY, OH, OK, OR, PA, SC, SD, TN, TX, UT, VT, WI
2012	IN, MO, MT, NC, ND, NH, WA, WV
2014	AL, AR, AZ, CO, CT, FL, GA, HI, IA, ID, IL, KS, MA, MD, ME, MI, MN, NE, NH, NM, NY, OH, OK, OR, PA, SC, TX, WI

House Elections in our Sample

Year	State-District
2000	AL-4, AR-4, CA-20, CA-49, CO-6, CT-5, FL-12, FL-22, FL-8, GA-7, KS-3, KY-3, KY-6, MI-8, MN-6, MO-2, MO-3, MO-6, NC-11, NC-8, NH-1, NH-2, NM-1, NV-1, OH-1, OH-12, OK-2, PA-10, PA-13, PA-4, TX-25, UT-2, VA-2, WA-1, WA-5, WV-2
2002	AL-1, AL-3, AR-4, CT-5, FL-22, IA-1, IA-2, IA-3, IA-4, IL-19, IN-2, KS-3, KS-4, KY-3, ME-2, MI-9, MS-3, NH-1, NH-2, NM-1, NM-2, OK-4, PA-11, PA-17, SC-3, TX-11, UT-2, WV-2
2004	CA-20, CO-3, CT-2, CT-4, FL-13, GA-12, IA-3, IN-8, KS-3, KY-3, MO-5, MO-6, NC-11, NE-2, NM-1, NM-2, NV-3, NY-27, OK-2, OR-1, TX-17, WA-5, WV-2
2006	AZ-5, AZ-8, CO-4, CO-7, CT-2, CT-4, CT-5, FL-13, FL-22, GA-12, HI-2, IA-1, IA-3, ID-1, IL-6, IN-2, IN-8, IN-9, KY-2, KY-3, KY-4, MN-6, NC-11, NH-2, NM-1, NV-3, NY-20, NY-24, NY-25, NY-29, OH-1, OH-12, OH-15, OH-18, OR-5, PA-10, SC-5, TX-17, VA-2, VA-5, VT-1, WA-5, WI-8
2008	AK-1, AL-2, AL-3, AL-5, AZ-3, AZ-5, AZ-8, CA-11, CA-4, CO-4, CT-4, CT-5, FL-16, FL-24, FL-8, GA-8, ID-1, IL-10, IN-3, KY-2, KY-3, LA-4, LA-6, MD-1, MI-7, MO-6, NC-8, NH-1, NH-2, NM-1, NM-2, NV-2, NV-3, NY-20, NY-24, NY-25, NY-26, NY-29, OH-1, OH-15, PA-10, PA-11, SC-1, VA-2, VA-5, WI-8, WV-2
2010	AL-2, AL-5, AR-2, AZ-1, AZ-5, AZ-8, CA-20, CA-45, CO-3, CO-4, CT-4, CT-5, FL-2, FL-22, FL-24, FL-8, GA-12, GA-8, HI-1, IA-1, IA-2, IA-3, IN-2, IN-8, KS-4, KY-6, MA-1, MD-1, MD-2, MI-1, MI-3, MI-7, MI-9, MN-6, MO-3, MO-4, MO-8, MS-1, NC-2, NC-5, NC-8, NE-2, NH-1, NH-2, NM-1, NM-2, NV-3, NY-20, NY-23, NY-24, NY-25, OH-1, OH-12, OH-13, OH-15, OH-16, OH-9, OK-5, OR-3, OR-5, PA-10, PA-11, PA-4, SC-2, SC-5, SD-1, TN-1, TN-4, TN-8, TN-9, TX-17, VA-2, VA-5, VA-9, WA-2, WI-8, WV-3
2012	AZ-2, CA-10, CA-24, CA-3, CA-36, CA-52, CA-7, CA-9, CO-3, CO-6, CO-7, CT-5, FL-18, GA-12, HI-1, IA-1, IA-2, IA-3, IA-4, IL-12, IL-13, IL-17, IL-8, IN-2, IN-8, KY-6, MA-6, ME-2, MI-6, MN-6, MN-8, MT-1, NC-7, ND-1, NH-1, NH-2, NM-1, NV-3, NY-19, NY-21, NY-24, NY-25, NY-27, OH-16, OH-6, PA-12, RI-1, SD-1, TX-23, UT-4, VA-2, VA-5, WI-8, WV-3
2014	AR-2, AZ-1, AZ-2, CA-21, CA-36, CA-52, CA-7, CO-6, CT-5, FL-18, FL-2, FL-26, GA-12, HI-1, IA-1, IA-2, IA-3, IL-10, IL-12, IL-13, IL-17, IN-2, ME-2, MI-7, MN-7, MN-8, MT-1, ND-1, NE-2, NH-2, NM-2, NV-3, NY-19, NY-21, NY-23, NY-24, VA-10, VA-2

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